ccalgs

Descendants of nilpotent associative algebras and computation of coclass graphs for such algebras

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Chapter 1

Introduction

This package contains implementations of the algorithms described in [EM17]. Namely it contains a descendants algorithm for nilpotent associative algebras and a method to calculate the roots of maximal descendant trees in the coclass graph associated to the nilpotent associative F-algebras of coclass r. The implementation relies on the ModIsom package, especially the representation of nilpotent associative F-algebras by canonical forms, see [Eic11] for details.

1.1 Nilpotent associative algebras and canonical forms

For convenience a lot of the functions in this package are implemented for canonical forms and for nilpotent associative algebras. Only the functions taking canonical forms as input are documented. The corresponding functions for nilpotent associative algebras are named similar to the ones for canonical forms, replacing CanoForm by NAAlg. One can convert between the two types using the following functions.

1.1.1 NAAlgByCanoForm

▷ `NAAlgByCanoForm(C)` (operation)

Converts a canonical form `C` to a nilpotent associative algebra.

1.1.2 CanoFormByNAAlg

▷ `CanoFormByNAAlg(A)` (operation)

Converts a nilpotent associative algebra `A` to a canonical form.

Chapter 2

Descendants of nilpotent associative algebras

2.1 Immediate descendants

Let F be a finite field and let A and B be two nilpotent associative F-algebras. Then B is called an immediate descendant of A if A is isomorphic to $B/B^{cl(B)}$. The difference $dim(B) - dim(A)$ is called the stepsize of the descendant. A nilpotent associative F-algebra is called capable, if it has descendants. In [EM17] an algorithm is described to calculate these immediate descendants. This algorithm and the capability check is implemented in the following functions.

2.1.1 ImmediateDescendantsOfCanoForm (for a single stepsize)

▷ `ImmediateDescendantsOfCanoForm(C, s)` (operation)

Given a canonical form `C` this function returns the immediate descendants of `C` with stepsize `s`.

2.1.2 ImmediateDescendantsOfCanoForm (for a range of stepsizes)

▷ `ImmediateDescendantsOfCanoForm(C, slist)` (operation)

Given a canonical form `C` this function returns the immediate descendants of `C` with stepsizes in `slist`.

2.1.3 ImmediateDescendantsOfCanoForm (for all stepsizes)

▷ `ImmediateDescendantsOfCanoForm(C)` (operation)

Given a canonical form `C` this function returns the immediate descendants of `C` of all stepsizes.

2.1.4 IsCapableCanoForm

▷ `IsCapableCanoForm(C)` (property)

This returns whether the given canonical form `C` is capable, i.e. whether it has descendants.

2.2 Classifying nilpotent associative algebras by dimension

The descendant algorithm can be used for a classification of nilpotent associative F-algebras by dimension. This is used in the following functions.

2.2.1 CanoFormsByDimAndGens

▷ `CanoFormsByDimAndGens(F, d, n)` (operation)

Given a finite field `F`, a generator number `d` and a dimension `n`, this function calculates canonical forms for the `d`-generator nilpotent associative `F`-algebras of dimension `n`.

2.2.2 CanoFormsByDim

▷ `CanoFormsByDim(F, n)` (operation)

Given a finite field `F` and a dimension `n`, this function calculates canonical forms for the nilpotent associative `F`-algebras of dimension `n`.

2.3 Enumerating nilpotent associative algebras

Sometimes it will be enough to enumerate the number of immediate descendants or the number of nilpotent associative algebras of a given dimension. This is implemented in the following functions using orbit counting methods. However, note that the enumeration of n-dimensional algebras still requires the computation of the $(n-1)$-dimensional algebras.

2.3.1 NumberOfImmediateDescendantsOfCanoForm (for a single stepsize)

▷ `NumberOfImmediateDescendantsOfCanoForm(C, s)` (operation)

Given a canonical form `C` this returns the number of immediate descendants of `C` with stepsize `s`.

2.3.2 NumberOfImmediateDescendantsOfCanoForm (for a range of stepsizes)

▷ `NumberOfImmediateDescendantsOfCanoForm(C, slist)` (operation)

Given a canonical form `C` this returns the number of immediate descendants of `C` with stepsizes in `slist`.

2.3.3 NumberOfImmediateDescendantsOfCanoForm (for all stepsizes)

▷ `NumberOfImmediateDescendantsOfCanoForm(C)` (operation)

Given a canonical form `C` this returns the number of immediate descendants of `C` of all stepsizes.

2.3.4 NumberOfNAAlgsByDimAndGens

▷ `NumberOfNAAlgsByDimAndGens(F, d, n)` (operation)

Given a finite field `F`, a generator number `d` and a dimension `n`, this function returns the number of `d`-generator nilpotent associative `F`-algebras of dimension `n`.

2.3.5 NumberOfNAAlgsByDim

▷ `NumberOfNAAlgsByDim(F, n)` (operation)

Given a finite field `F` and a dimension `n`, this function returns the number of nilpotent associative `F`-algebras of dimension `n`.

Chapter 3

Coclass theory and coclass graphs

Recall that for a nilpotent associative algebra A its coclass is defined as $cc(A) = dim(A) - cl(A)$. Fixing a finite field F and a non-negative integer r one can define the associated coclass graph. Its vertices correspond one-to-one to the isomorphism types of nilpotent associative F-algebras of coclass r and there is a directed edge A to B if A is isomorphic to $B/B^{cl(B)}$. Such an algebra is called coclass settled, if all its descendants are of the same coclass as the algebra itself. The ccalgs package provides methods for calculating the coclass of a given nilpotent associative algebra and determining whether an algebra is coclass settled, i.e. whether all of its descendants have the same coclass as the algebra itself. Furthermore there are methods available for calculating finite parts of a coclass graph and plotting these finite parts using the XGAP package.

3.1 Calculating coclass and determine coclass settledness

3.1.1 CoclassOfCanoForm

▷ `CoclassOfCanoForm(C)` (attribute)

This returns the coclass of the given canonical form `C`.

3.1.2 IsCoclassSettledCanoForm

▷ `IsCoclassSettledCanoForm(C)` (attribute)

Determines whether the given canonical form `C` is coclass settled.

3.2 Calculating coclass graphs

We usually divide the calculation of a coclass graph in two steps. The first step is the calculation of the roots of the maximal descendant trees as described in [EM17], the second step is an iterated computation of stepsize 1 descendants. The following functions implement these two steps.

3.2.1 RootsOfCoclassGraph

▷ `RootsOfCoclassGraph(F, r)` (operation)

Given a finite field `F`, a coclass `r` this function calculates the roots of the maximal descendant trees of the associated coclass graph.

3.2.2 RootsOfCoclassGraphByRank

▷ `RootsOfCoclassGraphByRank(F, r, d)` (operation)

Given a finite field `F`, a coclass `r` and a generator number `d` this function calculates the roots with the given number of generators of the maximal descendant trees of the associated coclass graph.

3.2.3 DescendantTreeOfCanoForm

▷ `DescendantTreeOfCanoForm(C, l)` (operation)

Given a canonical form `C` for a nilpotent associative F-algebra, this iteratedly calculates `l` levels of descendants of the same coclass as `C`.

3.3 Plotting coclass graphs

This section contains functions for plotting the constructed coclass graphs and trees. Note that plotting requires the XGAP package, see [CN12]. The parameter P in all these functions must be a graphic poset created using the function GraphicPoset from the XGAP package. Once plotted it is also possible to recover the canonical forms or nilpotent associative algebras associated to a vertex. Currently the following functions are available.

3.3.1 PlotDescendantTreeOfCanoForm

▷ `PlotDescendantTreeOfCanoForm(C, d, P)` (operation)

Given a canonical form `C`, this plots the descendant tree of canonical forms of the same coclass as `C` up to dimension `d` into the GraphicPoset `P`.

3.3.2 PlotCoclassGraph

▷ `PlotCoclassGraph(F, r, d, P)` (operation)

Given a finite field `F` and a coclass `r`, this plots the associated coclass graph up to dimension `d` into the GraphicPoset `P`.

3.3.3 GetSelectedCanoForms

▷ `GetSelectedCanoForms(P)` (operation)

Returns a list of canonical forms corresponding to the selected vertices in the GraphicPoset `P`.

References

[CN12] Frank Celler and Max Neunhoeffer. Xgap 4.23, a gap4 package, available from http://www.gap-system.org/packages/xgap.html, 2012. 9

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