Institute of Scientific Computing
Technische Universität Braunschweig
Prof. Hermann Matthies

Summer Term 2019
15.04.2019

Partitioned Methods for Multifield Problems:
Assignment 1: Iterative Linear Solvers

Exercise 1: **(20 points)**

We consider the convergence of some iterative methods approximating the solution $\mathbf{u}$ of a linear problem $\mathbf{Au} = \mathbf{b}$, where $\mathbf{A} \in \mathbb{R}^{n\times n}$ and $\mathbf{u}, \mathbf{b} \in \mathbb{R}^n$. Iterative methods start with a given estimate for $\mathbf{u}$ and try to get better approximations in further iteration steps. The methods can be expressed by the iterative rule

$$\mathbf{u}^{k+1} = \mathbf{G}\mathbf{u}^k + \mathbf{g}.$$

For the so called *Jacobi method* we get

$$\mathbf{G} := (\mathbf{I} - \mathbf{D}^{-1}\mathbf{A}), \qquad \mathbf{g} := \mathbf{D}^{-1}\mathbf{b},$$

for the *Successive Over-Relaxation method* (*SOR*) we get

$$\mathbf{G} := (\mathbf{D} + \omega\mathbf{L})^{-1}((1-\omega)\mathbf{D} - \omega\mathbf{U}), \qquad \mathbf{g} := (\mathbf{D} + \omega\mathbf{L})^{-1}\omega\mathbf{b},$$

with the unit matrix $\mathbf{I}$, and the decomposition $\mathbf{A} = \mathbf{D} + \mathbf{L} + \mathbf{U}$, where $\mathbf{D}$ is the diagonal matrix, $\mathbf{L}$ the strictly lower triangular matrix, and $\mathbf{U}$ the strictly upper triangular matrix of $\mathbf{A}$. ω marks the so called *relaxation factor* with $0 < \omega < 2$.

To get a feeling onto the convergence behaviour of the methods, we consider the linear system

$$\begin{pmatrix} 1 & a & a \\ a & 1 & a \\ a & a & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \tag{1}$$

with $a \in [-0.5, \dots, 1.0]$.

(a) **Spectral Radius.** Write a Matlab program to compute the spectral radius for the Jacobi and the SOR methods regarding the given coefficient matrix; choose $a \in \{-0.5, 0.0, 0.5, 0.8, 0.9, 1.0\}$. For the Jacobi method, plot the spectral radius over a; for the SOR method, plot the spectral radius over ω and find the optimal relaxation factor for each realisation of a quantitatively. (8 points)

(b) **Convergence.** The Jacobi and SOR implementations are provided by MATLAB functions. Display their convergence by plotting the relative error

$$\frac{||\mathbf{u}^k - \mathbf{u}||_2}{||\mathbf{u}||_2}$$

over the number of performed iterations for a realisation of a, for instance $a = 0.5$ (the reference solution $\mathbf{u}$ can be obtained through the MATLAB direct solver provided by operator \). For the SOR method do the plotting for four reasonable values of ω and confirm the results of the spectral radius computations quantitatively. (8 points)

(c) **BiCGStab.** The *BiCGStab method* is a non-linear iterative method to approximate the solution of a linear system. Use the MATLAB built-in *"bicgstab()"* function to implement the method and plot the relative error over the number of iterations. What can you say about its convergence? (4 points)

Exercise 2: **(16 points)**

Let matrice $\mathbf{A}_{11}$, $\mathbf{A}_{12}$, $\mathbf{A}_{21}$, $\mathbf{A}_{22}$ and vectors $\mathbf{f}_1$, $\mathbf{f}_2$ be as given in the provided downloadable matlab file, consider the two coupled problems:

$$P1: \quad \mathbf{A}_{11}\mathbf{x}_1 = \mathbf{f}_1 - \mathbf{A}_{12}\mathbf{x}_2$$

$$P2: \quad \mathbf{A}_{22}\mathbf{x}_2 = \mathbf{f}_2 - \mathbf{A}_{21}\mathbf{x}_1$$

Which can be written in a compact form

$$\mathbf{A}\mathbf{x} = \mathbf{f}.$$

Suppose we have only the direct solvers for P1 and P2 but have no direct solver for the global problem $\mathbf{A}\mathbf{x} = \mathbf{f}$.

(a) Write a Matlab program to solve $\mathbf{A}\mathbf{x} = \mathbf{f}$ by using iterative methods of block Jacobi and block Gauss-Seidel (in which the matrix $\mathbf{A}$ is partitioned into block diagonal matrix $\mathbf{D} = [\mathbf{A}_{11}, \mathbf{0}; \mathbf{0}, \mathbf{A}_{22}]$, block lower matrix $\mathbf{L} = [\mathbf{0}, \mathbf{0}; \mathbf{A}_{21}, \mathbf{0}]$ and block upper matrix $\mathbf{U} = [\mathbf{0}, \mathbf{A}_{12}; \mathbf{0}, \mathbf{0}]$). Compare their convergence speeds. (8 points)

(b) Let $\mathbf{A}_\alpha = \mathbf{A}[1:4,\ 1:4]$(the upper left 4-by-4 submatrix) and $\mathbf{A}_\beta = \mathbf{A}[5:7,\ 5:7]$ (the lower right 3-by-3 submatrix), and $\mathbf{B} = \begin{bmatrix} \mathbf{A}_\alpha & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_\beta \end{bmatrix}$, solve the preconditioned problem

$$\mathbf{B}^{-1}\mathbf{A}\mathbf{x} = \mathbf{B}^{-1}\mathbf{f}$$

by block Jacobi method and block Gauss-Seidel method, compare their convergence speeds with those in the original problem. (8 points)