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Cartoon-Texture-Noise Decomposition with Transport Norms

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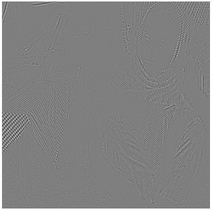
Problem

- Task: Decompose an observed image u^0 into a *cartoon part* u, a *texture part* v and a *noise part* w such that $u + v + w = u^0$.

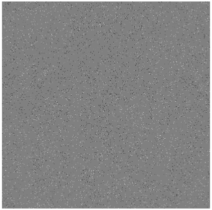
u^0

u

v



w



General Variational Approach

- Let $\Omega \subset \mathbb{R}^2$ be the *image domain* and $u^0 : \Omega \to \mathbb{R}$.
- Solve the problem

$$\min_{u,v} \quad \alpha F_u(u) + \beta F_v(v) + \gamma F_w(u^0 - u - v)$$

with positive constants α, β, γ and appropriate functionals F_u, F_v, F_w which capture discriminating features of cartoon, texture and noise.

Rudin/Osher/Fatemi Model [1992]

- The problem

$$\min_{u \in \mathrm{BV}(\Omega)} \quad \alpha \mathrm{TV}(u) + \frac{\beta}{2} \left\| u^0 - u \right\|_{L^2}^2$$

yields a decomposition into two components.

- Meyer: The ROF model does not capture texture properly.

Meyer Model [2001]

- Meyer's *G-Norm*:

$$G(\Omega) = \{v \in L^2(\Omega) \mid \exists g \in L^\infty(\Omega, \mathbb{R}^2) : \operatorname{div} g = v\}$$
$$\|v\|_G = \inf \{\| |g| \|_{L^\infty} \mid \operatorname{div} g = v\}$$

- The problem

$$\min_{(u,v) \in \mathrm{BV}(\Omega) \times G(\Omega)} \quad \alpha \mathrm{TV}(u) + \beta \|v\|_G \quad \text{s.t.} \quad u + v = u^0$$

separates cartoon and texture properly.

- There is still no third component that allows to discriminate texture and noise.

Vese/Osher Model [2003]

- Reformulation of Meyer's model:

$$\min_{(u,g)\in \mathrm{BV}(\Omega)\times L^\infty(\Omega,\mathbb{R}^2)} \alpha \mathrm{TV}(u) + \beta \left\| |g| \right\|_{L^\infty} \quad \text{s. t.} \quad u + \operatorname{div} g = u^0$$

- The problem

$$\min_{(u,g)\in \mathrm{BV}(\Omega)\times L^p(\Omega,\mathbb{R}^2)} \alpha \mathrm{TV}(u) + \frac{\beta}{p} \left\| |g| \right\|_{L^p}^p + \frac{\gamma}{2} \left\| u^0 - u - \operatorname{div} g \right\|_{L^2}^2$$

  approximates Meyer's G-Norm and relaxes the equality constraint.
- It allows for a decomposition into three components!

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Discriminating Features of Texture and Noise

- Texture features *oscillations* in the sense that local averages are close to zero, especially the total positive mass and the total negative mass are almost equal.
- *Gaussian noise* has a similar characteristic. Hence, the separation of texture and Gaussian noise is inherently difficult.
- We focus on *impulsive noise*: The total positive mass is almost equal to the total negative mass but local averages are in general not close to zero.
- Idea: One can move the positive and negative mass around to cancel each other out. This is cheap for texture and expensive for impulsive noise.

Transport Problem in Kantorovich Form [1942]

- Let μ, ν be measures on Ω with equal mass and $c : \Omega \times \Omega \to \mathbb{R}_+ \cup \{0\}$. Then,

$$\inf_{\pi} \left\{ \int_{\Omega\times\Omega} c(x,y) \, \mathrm{d}\,\pi(x,y) \mid \mathrm{proj}_1 \pi = \mu, \; \mathrm{proj}_2 \pi = \nu \right\}$$

is the *minimal cost to transport* μ *to* ν.

Wasserstein Metric [1969]

- In case $c(x,y) = d(x,y)^p$ for some metric d on Ω and $p \geq 1$,

$$W_p(\mu,\nu) = \inf_{\pi} \left\{ \int_{\Omega\times\Omega} d(x,y)^p \,\mathrm{d}\,\pi(x,y) \mid \mathrm{proj}_1\,\pi = \mu, \mathrm{proj}_2\,\pi = \nu \right\}^{\frac{1}{p}}$$

  is a metric on the space of probability measures.
- *Kantorovich-Rubinstein duality*:

$$W_1(\mu,\nu) = \sup_{f} \left\{ \int_{\Omega} f \,\mathrm{d}(\mu-\nu) \mid \mathrm{Lip}(f) \leq 1 \right\}$$

- $W_1(\mu,\nu)$ is infinite in case μ and ν have different total mass.

Kantorovich-Rubinstein Norm [2014]

- A variant with finite values for measures with different total mass is

$$\|\mu - \nu\|_{\mathrm{KR},\beta,\gamma} = \sup_f \left\{ \int_\Omega f \,\mathrm{d}(\mu - \nu) \mid \|f\|_{L^\infty} \leq \gamma,\ \||\nabla f|\|_{L^\infty} \leq \beta \right\}.$$

- Dualizing again, we obtain

$$\|\mu\|_{\mathrm{KR},\beta,\gamma} = \min_g \quad \gamma \|\mu - \operatorname{div} g\|_{L^1} + \beta \||g|\|_{L^1}.$$

- $\|\mu\|_{\mathrm{KR},\beta,\gamma} = \|\mu^+ - \mu^-\|_{\mathrm{KR},\beta,\gamma}$ is the cost to transport μ^+ to μ^- w.r.t. possible mass mismatch.

G'-Norm

- A dual formulation of Meyer's G-Norm is

$$\left\|u^0 - u\right\|_G = \sup_f \left\{ \int_\Omega f(u^0 - u)\,\mathrm{d}\,x \mid \left\| |\nabla f| \right\|_{L^1} \leq 1 \right\}.$$

- Repeating the step from W_1 to $\|\cdot\|_{\mathrm{KR},\beta,\gamma}$ leads to

$$\left\|u^0 - u\right\|_{G',\beta,\gamma} = \sup_f \left\{ \int_\Omega f(u^0 - u)\,\mathrm{d}\,x \mid \|f\|_{L^\infty} \leq \gamma,\ \left\| |\nabla f| \right\|_{L^1} \leq \beta \right\}.$$

- By duality,

$$\left\|u^0 - u\right\|_{G',\beta,\gamma} = \inf_g \quad \gamma \left\|u^0 - u - \operatorname{div} g\right\|_{L^1} + \beta \left\| |g| \right\|_{L^\infty}.$$

Decomposition with Transport Norms

- Meyer:

$$\min_{u,g} \quad \alpha \mathrm{TV}(u) + \beta \left\| |g| \right\|_{L^\infty} \quad \text{s. t.} \quad u + \operatorname{div} g = u^0$$

- Vese/Osher:

$$\min_{u,g} \quad \alpha \mathrm{TV}(u) + \frac{\beta}{p} \left\| |g| \right\|_{L^p}^p + \frac{\gamma}{2} \left\| u^0 - u - \operatorname{div} g \right\|_{L^2}^2$$

- New models:

$$\begin{aligned} &\min_{u} \quad \alpha \mathrm{TV}(u) + \left\| u^0 - u \right\|_{G',\beta,\gamma} \\ = &\min_{u,g} \quad \alpha \mathrm{TV}(u) + \beta \left\| |g| \right\|_{L^\infty} + \gamma \left\| u^0 - u - \operatorname{div} g \right\|_{L^1} \end{aligned}$$

$$\begin{aligned} &\min_{u} \quad \alpha \mathrm{TV}(u) + \left\| u^0 - u \right\|_{\mathrm{KR},\beta,\gamma} \\ = &\min_{u,g} \quad \alpha \mathrm{TV}(u) + \beta \left\| |g| \right\|_{L^1} + \gamma \left\| u^0 - u - \operatorname{div} g \right\|_{L^1} \end{aligned}$$

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Results

u^0

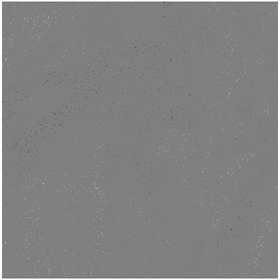
Perturbed Barbara image

Results

u

v

w



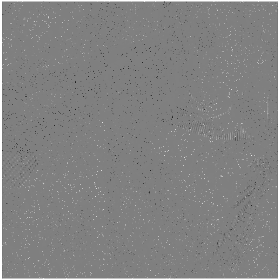
G'-norm: $\alpha = 1, \beta = 25000, \gamma = 1$

Results

u

v

w

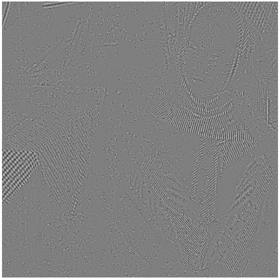


G'-norm: $\alpha = 1, \beta = 50000, \gamma = 1$

Results

u

v



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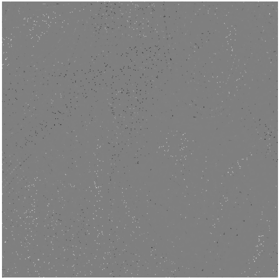
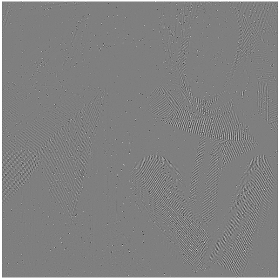


KR-norm: $\alpha = 1, \beta = 0.5, \gamma = 1$

Results

u

v



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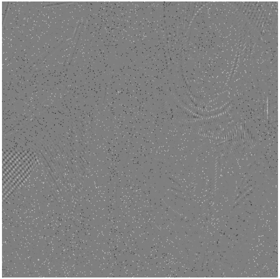


KR-norm: $\alpha = 1, \beta = 1, \gamma = 1$

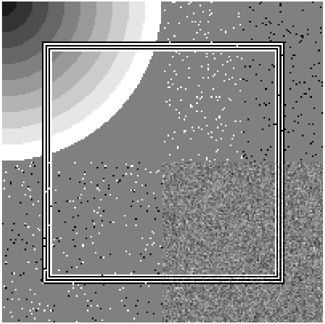
Sparsity in the Texture Part

$$\begin{aligned} &\min_{u} \quad \alpha \mathrm{TV}(u) + \left\| u^0 - u \right\|_{G',\beta,\gamma} \\ =&\min_{u,g} \quad \alpha \mathrm{TV}(u) + \beta \left\| |g| \right\|_{\mathbf{L}^\infty} + \gamma \left\| u^0 - u - \operatorname{div} g \right\|_{L^1} \end{aligned}$$

$$\begin{aligned} &\min_{u} \quad \alpha \mathrm{TV}(u) + \left\| u^0 - u \right\|_{\mathrm{KR},\beta,\gamma} \\ =&\min_{u,g} \quad \alpha \mathrm{TV}(u) + \beta \left\| |g| \right\|_{\mathbf{L}^1} + \gamma \left\| u^0 - u - \operatorname{div} g \right\|_{L^1} \end{aligned}$$

Results

u^0



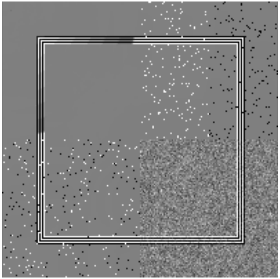
Artificial image

Results

u

v

w

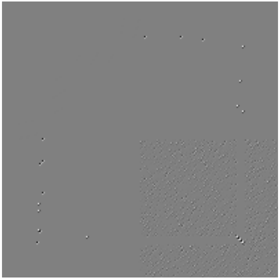


KR-norm: $\alpha = 2, \beta = 3, \gamma = 1$

Results

u

v



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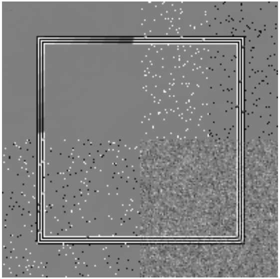
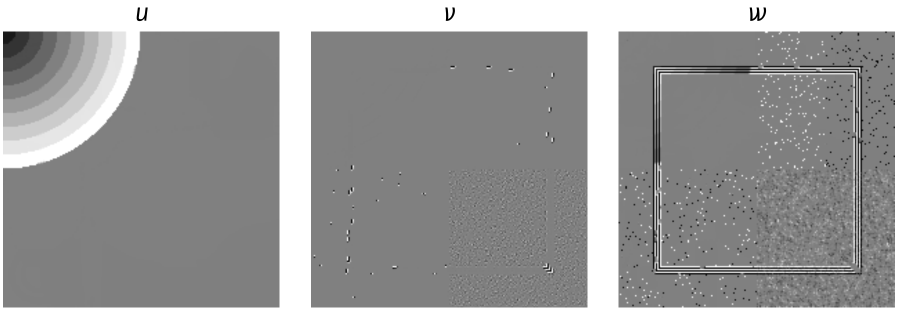


KR-norm: $\alpha = 2, \beta = 2.5, \gamma = 1$

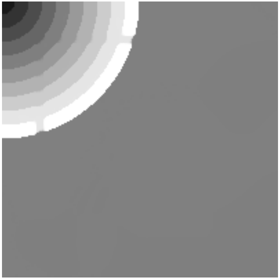
Results



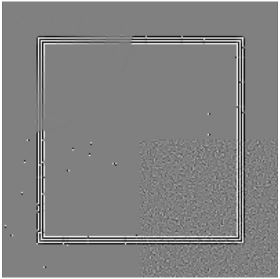
KR-norm: $\alpha = 2, \beta = 2, \gamma = 1$

Results

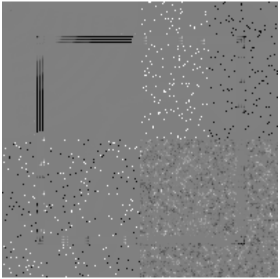
u



v

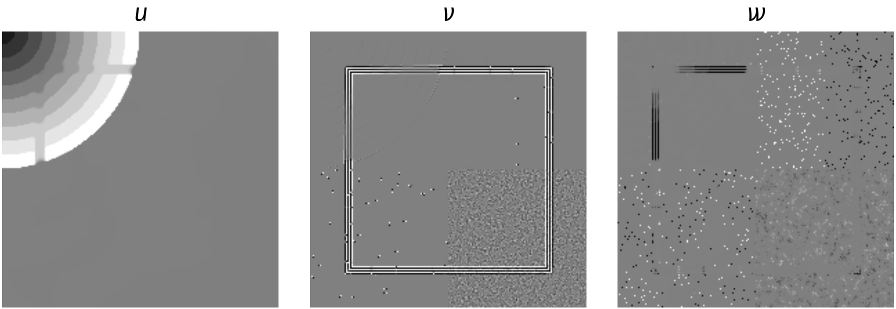


w



KR-norm: $\alpha = 2, \beta = 1.5, \gamma = 1$

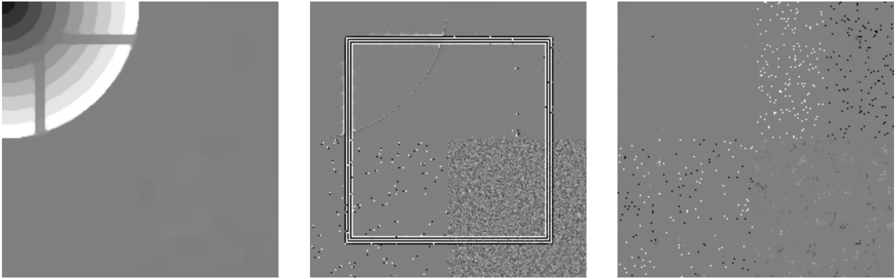
Results



KR-norm: $\alpha = 2, \beta = 1, \gamma = 1$

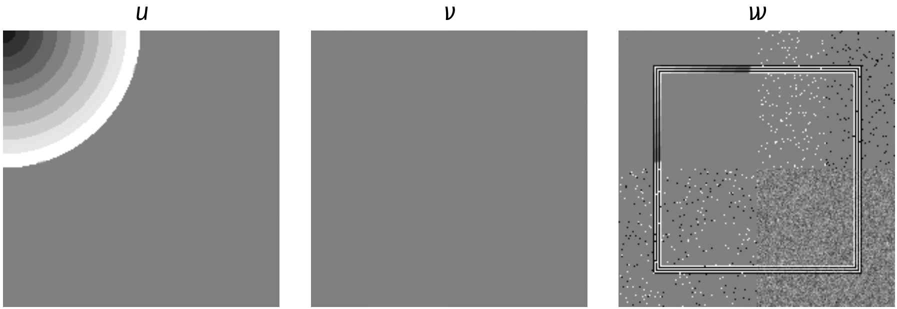
Results

u v w



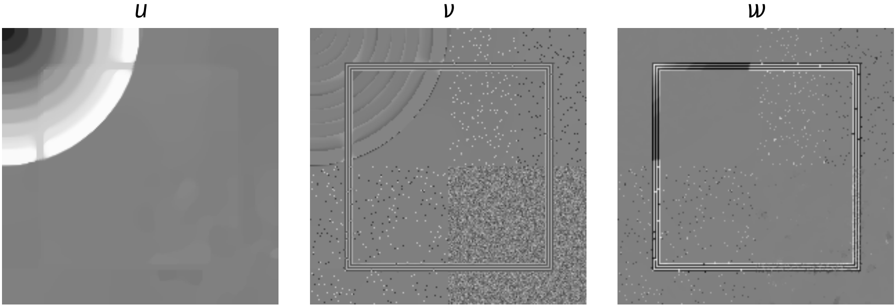
KR-norm: $\alpha = 2, \beta = 0.5, \gamma = 1$

Results



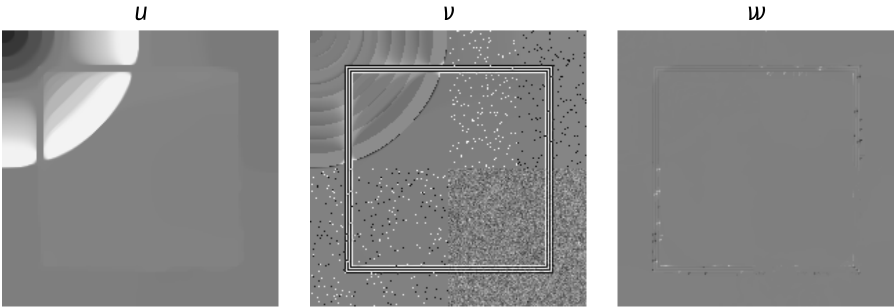
G'-norm: $\alpha = 2, \beta = 25000, \gamma = 1$

Results



G'-norm: $\alpha = 2, \beta = 10000, \gamma = 1$

Results



G'-norm: $\alpha = 2, \beta = 5000, \gamma = 1$

Conclusion

- There is a connection between image decomposition and optimal transportation.
- The separation of texture and Gaussian noise seems to be difficult.
- The separation of texture and impulsive noise can be handled using transport norms.

Thank you for your attention!