Institute of Scientific Computing
Technical University Braunschweig
Jaroslav Vondřejc, Ph.D.
Noemi Friedman, Ph.D.

Summer Term 2018
29.06.2018

Numerical Methods for PDEs (PDEs 2): Convergence, basis functions, FEM implementation and adaptivity

Exercise 1: *FEM: Triangulation properties, sparsity of the stiffness matrix* **(6 points)**

(a) Let d_T be the diameter of the largest circle contained in a triangle T. Prove that

$$\frac{d_T}{\text{diam}(T)} \leq \frac{1}{\sqrt{3}}$$

Solution

The ratio attains its maximum when the triangle is equilateral. For the equilateral triangle the diameter of the incircle is the third of the h height of the triangle

$$d_T = \frac{2h}{3}$$

If we assume that the edges have length a, then the hight is computed from

$$h^2 + \left(\frac{a}{2}\right)^2 = a^2,$$

that gives

$$h = \frac{\sqrt{3}}{2}a.$$

Then the diameter is

$$d_T = \frac{\sqrt{3}}{3}a = \frac{1}{\sqrt{3}}a.$$

The diameter of the triangle is the longest line that is contained in the triangle, which is the longest edge:

$$\text{diam}(T) = a.$$

The largest ratio is then

$$\frac{d_T}{\text{diam}(T)} = \frac{1}{\sqrt{3}}$$

– – – – – –

(3 points)

(b) What is the role of this measure in the convergence analysis?

Solution

Convergence properties are dependent on the quality of the triangulation. If the ratio is too small, then the triangles get very skinny, which deteriorates convergence. The error estimators are only shown for a non-degenerate family of triangulation, which means that the ratio can not get infinitely small during refinements.

– – – – – –

(1 points)

(c) A mesh of quadratic Lagrange triangles has two types of basis functions, those corresponding to vertex nodes and those corresponding to midpoint nodes. How many nodes are adjacent to a typical vertex node? To a typical midpoint node? How many nonzeros lie in each row of the corresponding stiffness matrix?

Solution

There are 8 adjacent nodes for every midpoint node, and the corresponding row of the stiffness matrix has 9 nonzero entries.
There are 18 adjacent nodes for every vertex node, and the corresponding row of the stiffness matrix has 19 nonzero entries.

– – – – – –

(2 points)

Exercise 2: *A priori error estimates* **(4 points)**

Suppose $\{T_h\}$ is a nondegenerate family of triangulations of a polygonal domain $\Omega \in \Re^2$ and suppose $f \in H^{d+1}(\Omega)$.
(a) If $f_I \in P_h^d$ is the piecewise polynomial interpolant of degree d of f, what can we know about the bound of the L^2 norm of the interpolation error. Please explain every expression in the bound.

Solution

$$\|f - f_I\|_{L^2(\Omega)} \leq Ch^{d+1}|f|_{H^{d+1}(\Omega)}$$

with

- C: positive constant
- $|\cdot|_{H^{d+1}(\Omega)}$: the seminorm

$$|f|_{H^{d+1}(\Omega)} = \sum_{i+j=d+1} \int_\Omega \left| \frac{\partial^{i+j} f}{\partial x^i \partial y^j} \right|^2 dxdy$$

- h: the maximum diameter of any triangle in T_h

– – – – – –

(2 points)

(b) This error bound is used to show convergence of the Galerkin method. What is the connection between the interpolation error and the error of the Galerkin approximation?
(2 points)

$\boxed{\textit{Solution}}$

We know from Cea's theorem that when the bilinear operator is bounded and v-elliptic then the error of the Galerkin approximation f_h is bounded:

$$\|f - f_h\|_V \leq \frac{M}{\delta}\|f - v_h\|_V \quad \forall v_h \in V_h$$

Here the contants M and δ are from the constants from the condition of boundedness and V-ellipticity, respectively. Since the piecewise polynomial interpolant u_I of u belongs to V_h, the above bound of the interpolation error will also give a bound for the error of the Galerkin approximation:

$$\|f - f_h\|_{L^2(\Omega)} \leq \frac{M}{\delta}\|f - f_I\|_{L^2(\Omega)} \leq \frac{M}{\delta}Ch^{d+1}|f|_{H^{d+1}(\Omega)}$$

$- - - - - -$

Exercise 3: C_1 *basis functions* **(7 points)**

Let us define piece-wise Hermite polynomials, to span the space of $C^1(\Omega)$ piece-wise polynomials of total degree d.

Suppose $\Omega \subset R$, so a 1D domain. Define the local basis functions defined over one element in its local coordinate system $\Phi_i(\xi) : \Omega_e \to \mathbb{R}, \quad \Omega_e = [-1, 1]$ as the linear combination of the monomials:

$$\phi_i(\xi) = a_{i0} + a_{i1}\xi + a_{i1}\xi^2 + \ldots + a_{id}\xi^d.$$

The vector of coefficients

$$a_i^T = \begin{bmatrix} a_{i0} & a_{i1} & \ldots & a_{id} \end{bmatrix}$$

can be calculated from i linear system of equations:

$$\tilde{B}a_i = b_i.$$

What is the matrix $\tilde{B}$ and what are the right hand sides b_i when a two-node element is used? The two nodes are at the coordinates $\xi_1 = -1$ and $\xi_2 = 1$. Draw a draft of all the basis functions defined over the element.

Hint:

- Use $N = 4$ and $d = 3$ to have a well defined system of equations.

Draft of basis functions $\{\phi_i\}_{i=1}^N$:

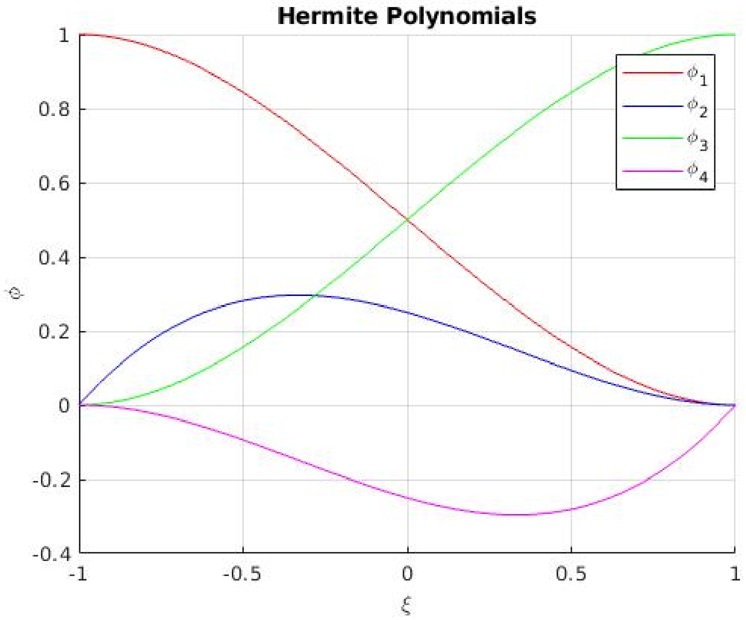
Determination of matrix $\tilde{B}$:

$\boxed{\textit{Solution}}$

The constraints for the basis functions are

$$\begin{aligned}
\phi_1(\xi_1) &= 1 \\
\phi_1'(\xi_1) = \phi_1(\xi_2) = \phi_1'(\xi_2) &= 0 \\
\phi_2'(\xi_1) &= 1 \\
\phi_2(\xi_1) = \phi_2(\xi_2) = \phi_2'(\xi_2) &= 0 \\
\phi_3(\xi_2) &= 1 \\
\phi_3(\xi_1) = \phi_3'(\xi_1) = \phi_3'(\xi_2) &= 0 \\
\phi_4'(\xi_2) &= 1 \\
\phi_4(\xi_1) = \phi_4'(\xi_1) = \phi_4(\xi_2) &= 0 \\
&.
\end{aligned}$$

The basis functions satisfying these constraints are shown below.



Using the polynomial form of the basis functions the function and their derivatives evaluated at the ξ_1 and ξ_2 points read

$$\begin{bmatrix}\phi_i(\xi_1)\\ \phi_i'(\xi_1)\\ \phi_i(\xi_2)\\ \phi_i'(\xi_2)\end{bmatrix} = \begin{bmatrix}1 & \xi_1 & \xi_1^2 & \xi_1^3\\ 0 & 1 & 2\xi_1 & 3\xi_1^2\\ 1 & \xi_2 & \xi_2^2 & \xi_2^3\\ 0 & 1 & 2\xi_2 & 3\xi_2^2\end{bmatrix}\begin{bmatrix}a_{i,0}\\ a_{i,1}\\ a_{i,2}\\ a_{i,3}\end{bmatrix} = \begin{bmatrix}1 & -1 & 1 & -1\\ 0 & 1 & -2 & 3\\ 1 & 1 & 1 & 1\\ 0 & 1 & 2 & 3\end{bmatrix}\begin{bmatrix}a_{i,0}\\ a_{i,1}\\ a_{i,2}\\ a_{i,3}\end{bmatrix}$$

The first four constraints then are given by the system of equations:

$$\begin{bmatrix}\phi_1(\xi_1)\\ \phi_1'(\xi_1)\\ \phi_1(\xi_2)\\ \phi_1'(\xi_2)\end{bmatrix}\begin{bmatrix}1 & -1 & 1 & -1\\ 0 & 1 & -2 & 3\\ 1 & 1 & 1 & 1\\ 0 & 1 & 2 & 3\end{bmatrix}\begin{bmatrix}a_{1,0}\\ a_{1,1}\\ a_{1,2}\\ a_{1,3}\end{bmatrix} = \begin{bmatrix}1\\ 0\\ 0\\ 0\end{bmatrix}$$

In general, the coefficients of the ith basis functions can be solved from the system of equations

$$\tilde{B}a_i = b_i,$$

where

$$\tilde{B} = \begin{bmatrix}1 & -1 & 1 & -1\\ 0 & 1 & -2 & 3\\ 1 & 1 & 1 & 1\\ 0 & 1 & 2 & 3\end{bmatrix}$$

and

$$b_i = \begin{bmatrix}b_{i,0}\\ b_{i,1}\\ b_{i,2}\\ b_{i,3}\end{bmatrix}$$

with

$$b_{i,j} = \begin{matrix}1 & \text{if} i = j\\ 0 & \text{otherwise}\end{matrix}$$

$- - - - --$

Exercise 4: C_0 *basis functions* **(7 points)**

Now we define piece-wise Lagrange polynomials to span the space of all $C^0(\Omega)$ piece-wise polynomials of total degree d. Let $\Omega = (0,1) \times (0,1)$ be a unit square and consider a uniform triangulation of Ω created by dividing Ω into n^2 sub-squares, each with side length $h = 1/n$, and then dividing each sub-square into two triangles. Consider two different triangulations:

1. $\{T_1\}$: uniform linear Lagrange triangulation with $n = 2k$ (with $2(2k)^2$ triangles, $d = 1$);
2. $\{T_2\}$: uniform quadratic Lagrange triangulation with $n = k$ (with $2k^2$ triangles, $d = 2$);

- How many nodes are needed per inner edges in the triangulations $\{T_1\}$ and $\{T_2\}$ and why?

  Solution

  For assuring the continuity of a d degree function one needs $d + 1$ number of points per inner edges. That means 2 nodes for $\{T_1\}$ and 3 for $\{T_2\}$.

  – – – – – –

- Which triangulation has more nodes (you can try for example $k = 4$, and draw a draft of the mesh)?

  Solution

  They have the same number of nodes. The stiffness matrix is of the same size.

  – – – – – –

- Which resulting stiffness matrix: $K_{ij} = \int_\Omega \nabla\phi_i \cdot \nabla\phi_j$ has more nonzero elements (which one is more sparse) and why? What is the maximum number of nonzero elements per row for the two triangulations?

  Solution

  The stiffness matrix for $\{T_1\}$ is sparser because the nodes have more adjacent nodes in $\{T_2\}$. $\{T_1\}$ has maximum 7 nz elements per row and $\{T_2\}$ has max 19.

  – – – – – –

- Draw one triangle and its nodes that has to be used when $d = 4$. Explain the number of the nodes.

Solution

The general polynomial take the form

$$p = a_1 + a_2x + a_3y + a_4x^2 + a_5xy + a_6y^2 + \ldots + a_{15}y^4$$

which can be well defined by 15 coefficients, so 15 nodes are needed. The number of the nodes to assure continuity at the edge is $d + 1 = 5$.

– – – – – –

Exercise 5: *Numerical integration over quadrilateral element* **(7 points)**

Assume the domain $\Omega = (-1, 1) \times (-1, 1)$. Now the task is to compute the integral

$$\int\int_\Omega (x^3y + x^2y + 3y + 5)\mathrm{d}x\mathrm{d}y \tag{1}$$

with Gauß quadrature.

(a) Which point rule has to be used in the x direction and which in the y direction to get exact solution of (1)? (Give the minimum number of the points to be used for the two univariate rules.) Explain the answer in one sentence.

Solution

The polynomial can be written as linear combination of the tensor product of the monomials: $\{1, x, x^2, x^3\}$ and the monomials $\{1, y\}$.

- in the x direction: $2n_x - 1 = d_x = 3 \rightarrow n_x = 2$
- in the y direction: $2n_y - 1 = d_y = 1 \rightarrow n_y = 1$

– – – – – –

(b) Collect in a table the coordinates of the integration points and the corresponding weights of the combined rule to be used to calculate the given integral. You can use the points and the weights for the univariate rule in Table 1.

Solution

i	x_i	y_i	ω_{1x}	ω_{2x}	ω_i	$f(x_i, y_i)$
1	$\sqrt{\frac{1}{3}}$	0	1	2	2	5
2	$-\sqrt{\frac{1}{3}}$	0	1	2	2	5

– – – – – –

(c) Calculate the integral.

Solution

$$\int\int_\Omega (x^3y + x^2y + 3y + 5)\mathrm{d}x\mathrm{d}y = f(x_1, y_1)\omega_1 + f(x_2, y_2)\omega_2 = 20$$

– – – – – –

Exercise 6: *Mesh generation, adaptivity, a posterior error estimates* **(6 points)**

(a) What does it mean nonconforming triangulation? Draw an example.

number of points, n	Points, x_i	Weights, w_i
1	0	2
2	$\pm\sqrt{\frac{1}{3}}$	1
3	0	$\frac{8}{9}$
	$\pm\sqrt{\frac{3}{5}}$	$\frac{5}{9}$
4	$\pm\sqrt{\frac{3}{7}-\frac{2}{7}\sqrt{\frac{6}{5}}}$	$\frac{18+\sqrt{30}}{36}$
	$\pm\sqrt{\frac{3}{7}+\frac{2}{7}\sqrt{\frac{6}{5}}}$	$\frac{18-\sqrt{30}}{36}$
5	0	$\frac{128}{225}$
	$\pm\frac{1}{3}\sqrt{5-2\sqrt{\frac{10}{7}}}$	$\frac{322+13\sqrt{70}}{900}$
	$\pm\frac{1}{3}\sqrt{5+2\sqrt{\frac{10}{7}}}$	$\frac{322-13\sqrt{70}}{900}$

Table 1: Points and weights of the univariate Gauss-Legendre quadrature rule

Solution

That there is at least one hanging node, that is, node that belongs to one element, but does not belong to its neighboring element.

– – – – – –

(b) Explain the main idea of the strategy for choosing which triangles to refine due to Babuska and Rheinboldt by answering the following questions. Suppose there is a triangulation $\mathscr{T}_h$ with triangles $\{T_i\}_{i=1}^N$. The longest edge of the triangles is noted be h_i. Suppose that with an available error estimator we can compute the elementwise errors $\{\epsilon_i^{(1)}\}_{i=1}^N$. Let's suppose that this triangulation is coming from a uniform refinement, and we also computed the error estimators for the courser mesh, so we have for all the T_i triangles an error estimator $\{\epsilon_i^{(0)}\}_{i=1}^N$, which is the element-wise error for the triangle the subtriangles T_i were refined from.

- Main Assumptions: 1) What is our main assumption on the dependence of the elementwise error ϵ_i on the h_i diameter? 2) When do we call a mesh to be optimal?

  Solution

  1. That the it takes the form $\epsilon_i = ch_i^\lambda$, where c and λ are positive constants
  2. When the errors a equilibrated, that is when the elementwise errors a nearly constant

  – – – – – –

- How do we compute the constants in the first assumption?

  Solution

  According to the assumptions

$$\epsilon_i^0 = ch_i^{(0)\lambda}$$

and

$$\epsilon_i^1 = ch_i^{(1)\lambda},$$

from which the constants c and λ can be determined.

— — — — ——

- There is an important a-posteriori measure, which helps choosing triangles for refinement. How do we compute this measure?

  $\boxed{\textit{Solution}}$

  First a prediction of the error should be computed for each element if a further uniform refinement were carried out

  $$\epsilon_i^2 = c\left(\frac{h_i^{(1)}}{2}\right)^\lambda,$$

  Then the measure is the biggest error, that is

  $$M = \max\{\epsilon_i^2\}$$

— — — — ——

- Using the above mentioned measure, what is the criteria that chooses a triangle to be refined?

  $\boxed{\textit{Solution}}$

  Choose the i triangles for refinement, for which

  $$\epsilon_i^1 > M,$$

— — — — ——

(c) What are the three components to an adaptive algorithm? (Give only a concise definition for all three.)

$\boxed{\textit{Solution}}$

1. An element-by-element error estimator
2. a strategy for choosing which trangles to refine
3. an algorithm for locally refining a mesh.

— — — — ——