Technische Universität Braunschweig

Numerical methods for PDEs
FEM – convergence, a-priori error estimates, piecewise polynomials

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Contents of the course

- Fundamentals of functional analysis

- Abstract formulation FEM
- Spatial (meshing) and functional discretization (the basis functions)
- Convergence, regularity
- Variational crimes
- Implementation

- Error indicators/estimation
- Adaptivity

- Mixed formulations (e.g. Stokes)
- Stabilisation for flow problems

Abstract formulation, examples

- FEM and piecewise polynomials
- Non-degenerate triangulation, refinement of triangulation
- Convergence using piecewise linear functions
- Convergence using piecewise higher order elements
- Implementation, sparsity of the stiffnes matrix

- Variational crime: numerical integration
- Variational crime: curved boundaries

Recap: boundedness of the error of Galerkin method

$$a(u - u_h, v) = 0 \qquad \forall v \in V_h$$

If $a(\cdot,\cdot)$ is an inner product, it means that the approximation is an orthogonal projection to the subspace in the energy norm:

$$\text{error} = \|u(\mathbf{x}) - u_h\,(\mathbf{x})\|_E \leq \|u(\mathbf{x}) - v(\mathbf{x})\|_E \qquad \forall v(\mathbf{x}) \in V_h$$

According to Céa's theorem (see prove at the lecture note), even without $a(\cdot,\cdot)$ being symmetric, **the error of the approximation of Galerkin will be allways bounded**:

$$\|u - u_h\,\| \leq \frac{M}{\delta}\|u - v\| \qquad \forall v \in V_h$$

Where M and δ are constants from the conditions of boundedness and V-ellipticity of the bilinear term $a(\cdot,\cdot)$:

$$a(u, v) \leq M\|u\|\|v\|$$
$$a(u, u) \geq \delta\|u\|^2$$

Recap: what do we have to solve?

$$\underbrace{\sum_{i=1}^{N} c_i \int_\Omega \nabla\Psi_i(\mathbf{x}) \cdot \nabla\Psi_j(\mathbf{x})\, d\Omega}_{K_{ij}} = \underbrace{\int_\Omega f(\mathbf{x})\Psi_j(\mathbf{x})\, d\Omega}_{f_j} \Rightarrow \boxed{\mathbf{Kc} = \mathbf{f}}$$

FEM: Galerkin method where Ψ_j are piecewise polynomials

Main goals when implementing:
- efficient calculation of K
- efficient calculation of f
- solve Kc=f efficiently
- true solution is approximated well (error is small enough)

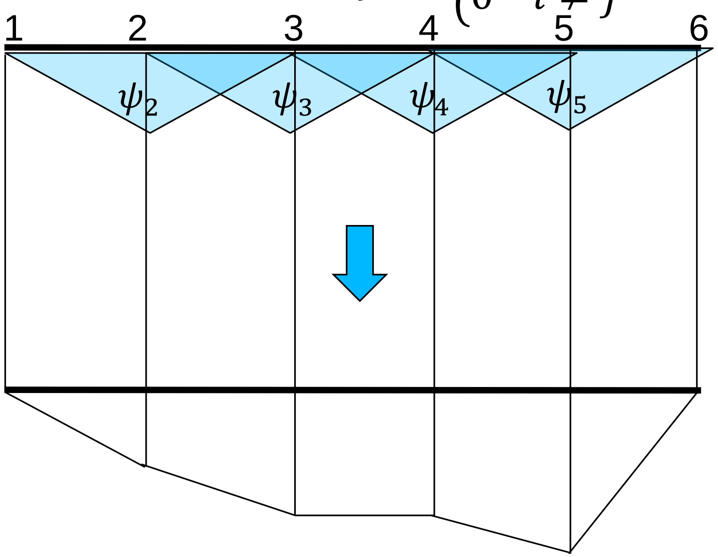
$\Rightarrow$ piecewise polynomials

Piecewise polynomials and the FEM

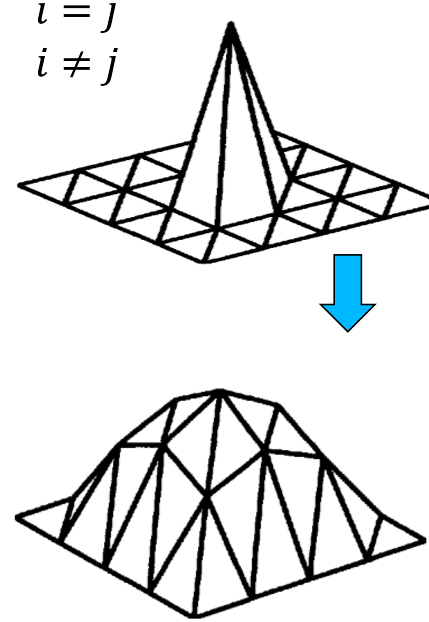
piecewise polynomials: a function that is defined by a polynomial on each subdomain
mesh: the collection of subdomains

nodal basis:

$$\psi_i(x_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$



$$\psi_i(x_j, y_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$



$$u(\mathbf{x}) \approx u_h(\mathbf{x}) = \sum_{i=1}^{4} u_i \, \psi_i(\mathbf{x})$$

$$u_h(\mathbf{x}) \in {P_h}^1$$

continious piecewise linear functions

for Poisson equation we have to show that ${P_h}^1 \subset H_1$
when it’s not satisfied it is not a ‚conforming element method’

Recap: Why linear piecewise polynomials defined on a triangulation are in the subspace

[Chapter 4.1.1]

Suppose, that uniqueness and existence can be shown in H_1

Show, that the space of all continious piecewise linear functions defined on the triangulation $\mathcal{T}_h$ is a subspace of H_1 .

The continious piecewise linear functions can be writen in the general form:

$$u = a_i + b_i x + c_i y \qquad (x, y) \in T_i$$

$$a_j + b_j x + c_j y = a_i + b_i x + c_i y \qquad \forall (x, y) \in e = T_i \cap T_j,$$

We have to show that all us are in H_1:

- $\int_\Omega u^2 < \infty \qquad \Omega = \cup_i T_i$
- $\int_\Omega \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 < \infty$

Recap: Why linear piecewise polynomials defined on a triangulation are in the subspace

1) First let's show that

$$\int_\Omega u^2 < \infty$$

$$\int_\Omega u^2 = \sum_i \int_{T_i} (a_i + b_i x + c_i y)^2 < \sum_i \int_{T_i} \max(a_i + b_i x + c_i y)^2 |T_i| < \infty$$

2.)Show that $\int_\Omega u^2 < \infty \qquad \Omega = \cup_i T_i$

There is no derivatives in a strong sense, but let's see whether these give weak derivatives:

$$g_1 := \frac{\partial u}{\partial x} = b_{\mathrm{i}} \qquad (x, y) \in \mathrm{int}(T_i)$$

$$g_2 := \frac{\partial u}{\partial y} = c_{\mathrm{i}} \qquad (x, y) \in \mathrm{int}(T_i)$$

Recap: Why linear piecewise polynomials defined on a triangulation are in the subspace

are weak derivatives if for any $v \in C_0^\infty$

$\int_\Omega u \frac{\partial v}{\partial x} = -\int_\Omega v g_1$ and $\int_\Omega u \frac{\partial v}{\partial y} = -\int_\Omega v g_2$ holds.

We prove only this, because the approach of the derivation for g_2 is same.

Let's first look at the left hand side of the equation:

$$\int_\Omega u \frac{\partial v}{\partial x} = \sum_i \int_{T_i} u \frac{\partial v}{\partial x} = \sum_i \left\{ \int_{\partial T_i} uv \cdot n_1 + \int_{T_i} \frac{\partial u}{\partial x} v \right\}$$

$\int_{\partial T_i} uv \cdot n_1 = 0$ on the edges of the boundary, and $\int_{\partial T_i} uv \cdot n_1 = -\int_{\partial T_j} uv \cdot n_1$

On the common edges of neighboring triangles T_i and T_j

$$\sum_i \left\{ \int_{\partial T_i} uv \cdot n_1 \right\} = 0$$

$$\int_{T_i} \frac{\partial u}{\partial x} v = \int_{T_i} b_i v$$

Recap: Why linear piecewise polynomials defined on a triangulation are in the subspace

$$\int_{\Omega} u \frac{\partial v}{\partial x} = -\int_{\Omega} v g_1$$

The left hand side of the equation:

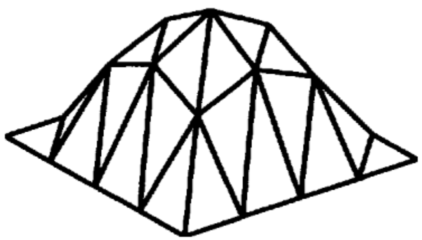
$$\int_{\Omega} u \frac{\partial v}{\partial x} = -\sum_i \left\{ \int_{T_i} b_i v \right\}$$

The left hand side of the equation:

$$-\int_{\Omega} v g_1 = -\sum_i \left\{ \int_{T_i} b_i v \right\}$$

Piecewise polynomials and the FEM

$v \in P_h^{(1)}$ ⟹ $v(x, y) = a_i + b_i x + c_i y \quad (x, y) \in T_i$



Derivatives in the classical sense:

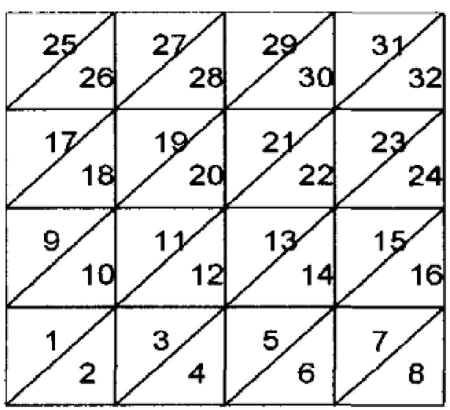
$$\frac{\partial v}{\partial x}(x, y) = b_i, \ (x, y) \in \mathrm{int}(T_i)$$
$$\frac{\partial v}{\partial y}(x, y) = c_i, \ (x, y) \in \mathrm{int}(T_i).$$

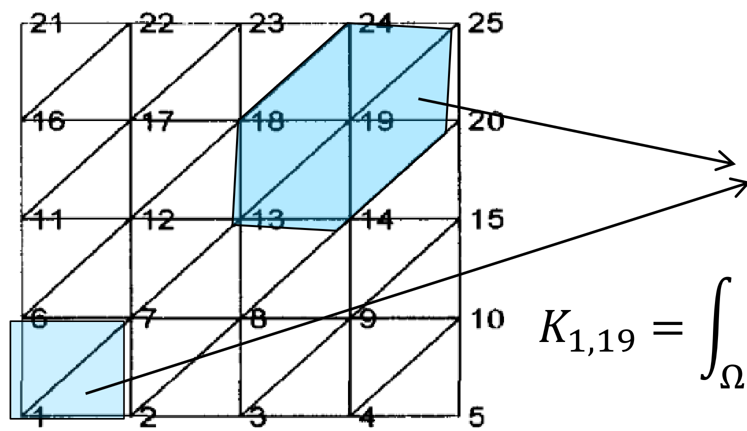
⟹ weak derivatives of v

(see proof in Gockenbach Chapter 4.1)

⟹ $v(x,y) \in L_2$, $v'(x,y) \in L_2$

⟹ ${P_h}^1 \subset H_1$





$\Psi_1, \Psi_{19} \neq 0$

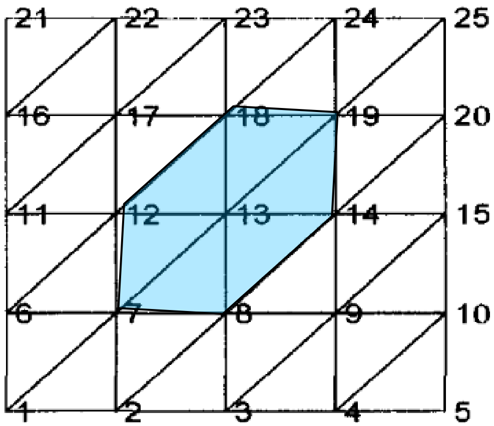
⟹

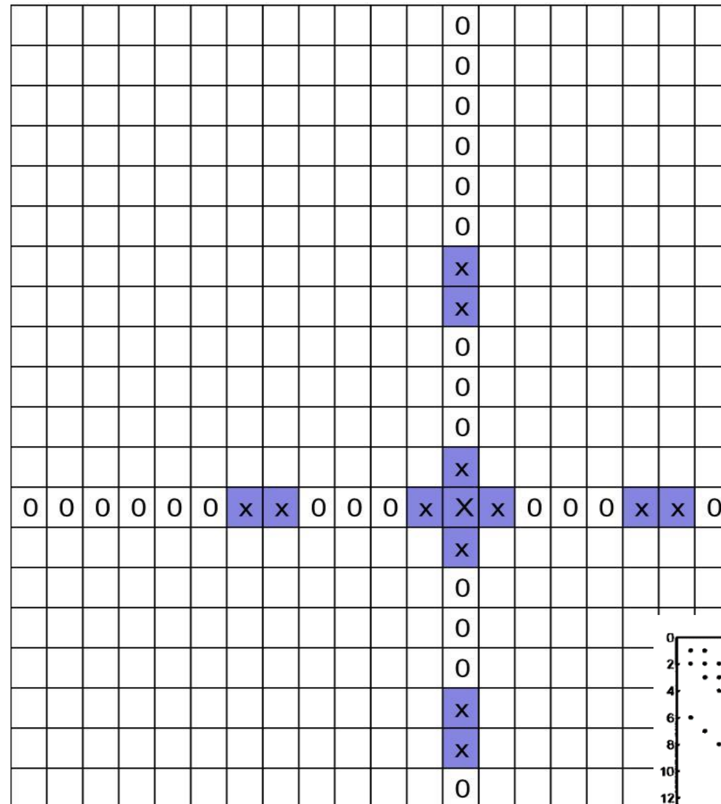
$$K_{1,19} = \int_\Omega \nabla\Psi_1(\mathbf{x}) \cdot \nabla\Psi_{19}(\mathbf{x})\, d\Omega = 0$$

Sparsity of the stiffness matrix

$$K_{ij} = \int_{\Omega} \nabla\Psi_i(\mathbf{x}) \cdot \nabla\Psi_j(\mathbf{x})\, d\Omega \neq 0$$ ⟹ if nodes i and j are adjacent

Example:





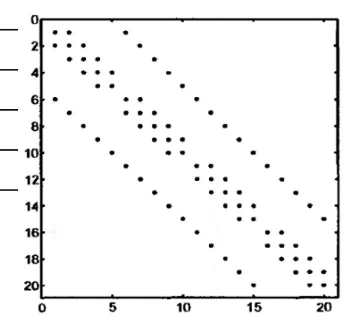
$K_{13,j} \neq 0$ if $j = 7,8,12,13,14,18,19$

Similarly:

$K_{i,13} \neq 0$ if $i = 7,8,12,13,14,18,19$

⇓

max 7 nonzero elements/row and /column



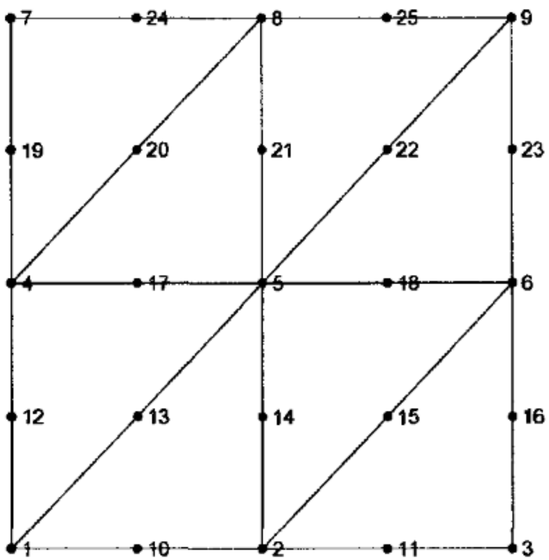
Quadratic piecewise polynomials

$\Psi_i(x, y) =$

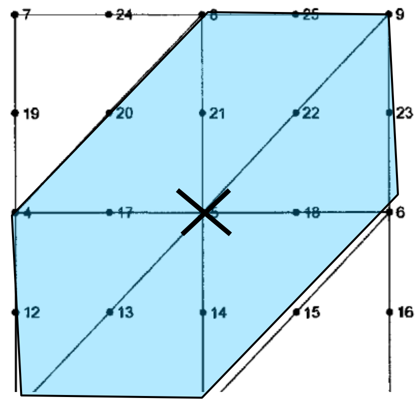
$a + bx + cy + dx^2 + exy + fy^2$

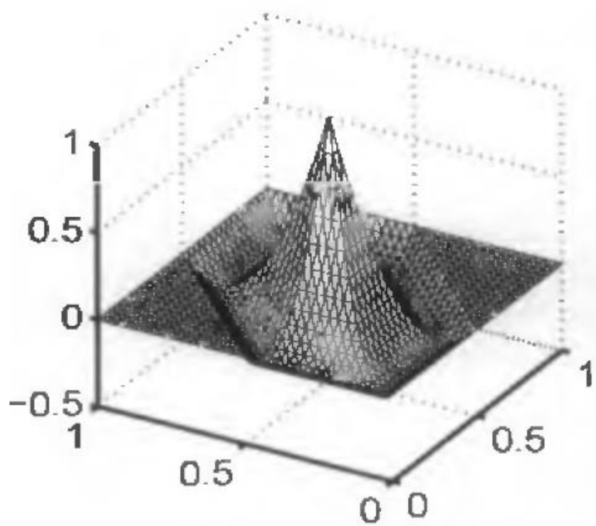
↓

6 nodes needed

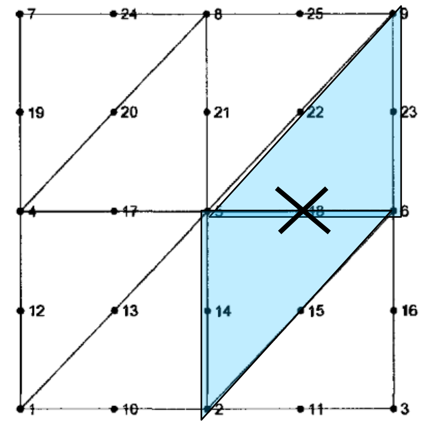


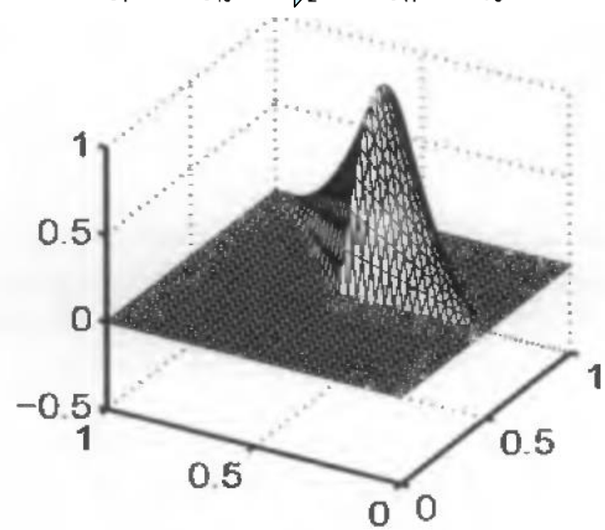
Ψ_5



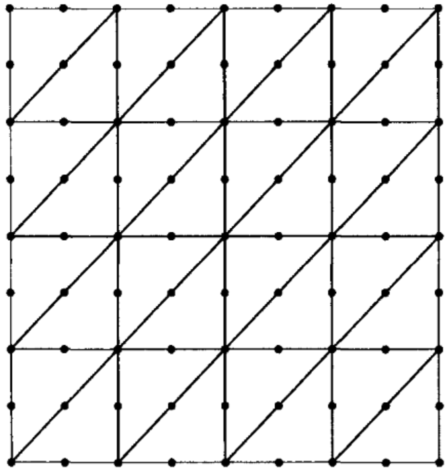


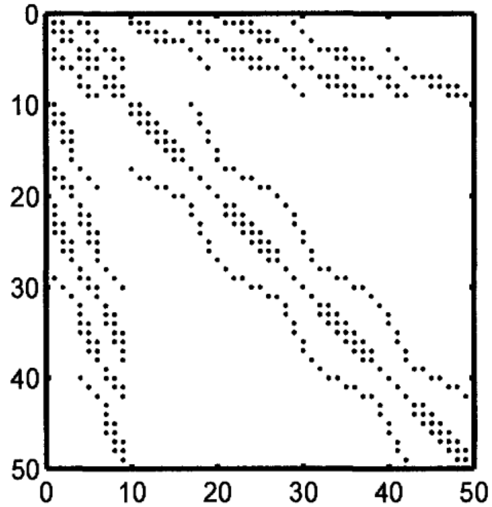
Ψ_{18}





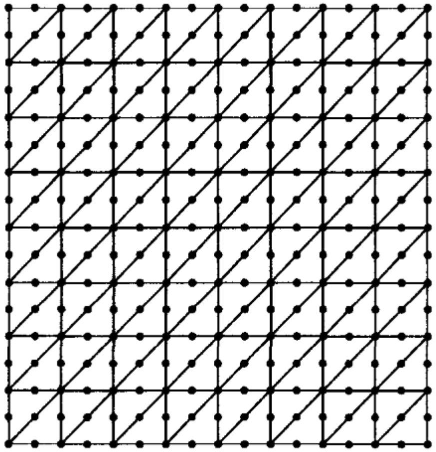
Quadratic piecewise polynomials

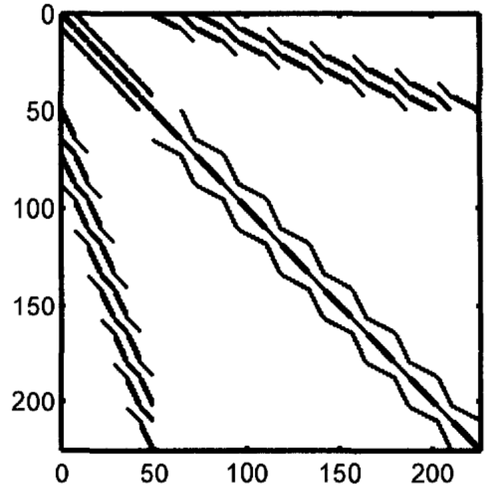




$nz = 405$

$$\frac{405}{50 \times 50} = 16\%$$



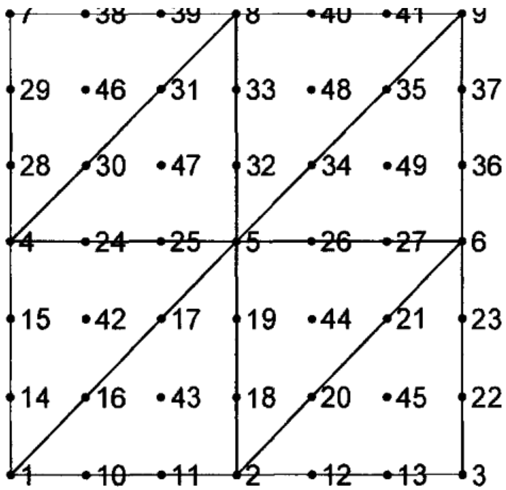


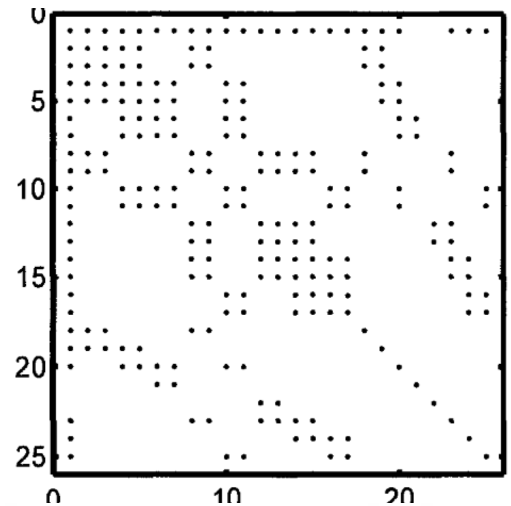
$nz = 2219$

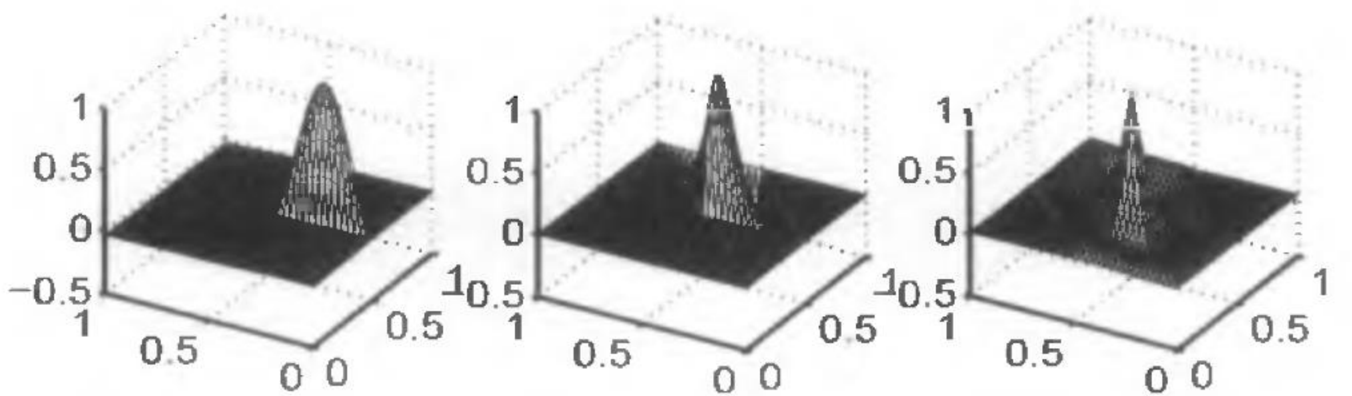
$$\frac{2219}{200 \times 200} = 5\%$$

Cubic piecewise polynomials

$$a + bx + cy + dx^2 + exy + fy^2 + gx^3 + hx^2y + ixy^2 + jy^3$$







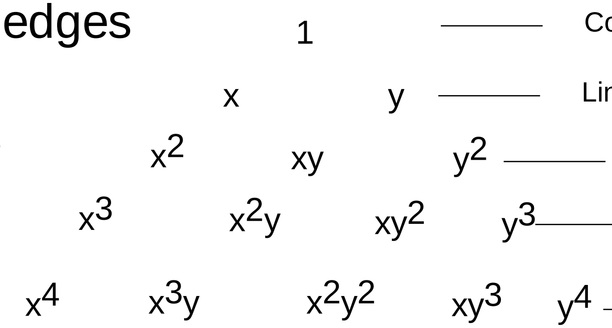
Higher order piecewise polynomials

Requirements for polynomial of degree d in 2D (two variables)

- number of nodes per edge (to guarantee continuity of the ansatz function):

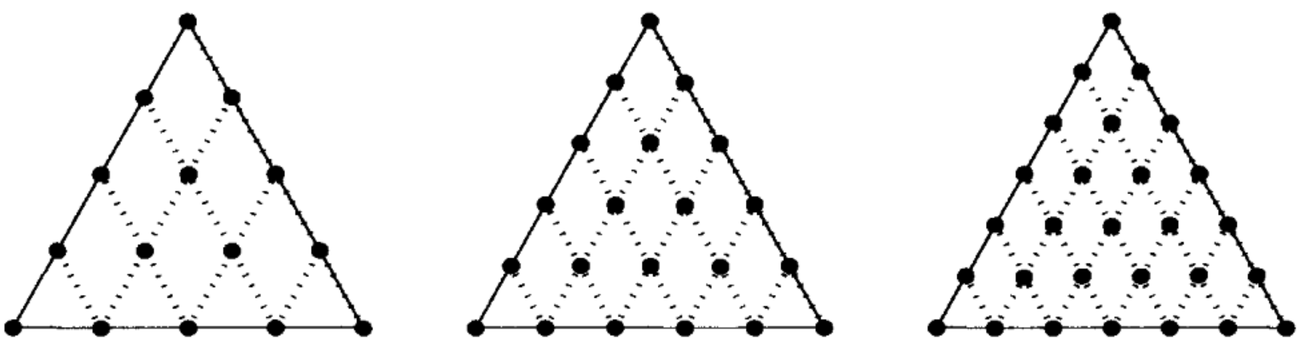
$$d+1$$

$$(d-1 \text{ in between the vertices}) \rightarrow 3d \text{ on the edges}$$



- Number of parameters needed to define 2D polynomials

$$1+2+\cdots+(d+1)=\frac{(d+1)(d+2)}{2}$$



$d=4$ $d=5$ $d=6$

Higher order piecewise polynomials

Let's check setup for $d = 4$ in 2D (two variables)

- number of nodes per edge (to guarantee continuity of the ansatz function):

$$d + 1 = 5$$

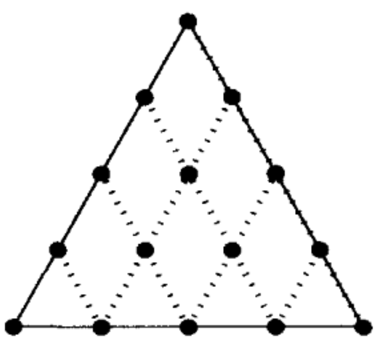
($d - 1$=3 in between the vertices) $\rightarrow 3d = 12$ on the edges

1 Co
x y Lin
x^2 xy y^2
x^3 x^2y xy^2 y^3
x^4 x^3y x^2y^2 xy^3 y^4

- Number of parameters needed to define 2D polynomials

$$1 + 2 + \cdots + (d+1) = \frac{(d+1)(d+2)}{2}$$

$$\frac{(d+1)(d+2)}{2} = 15$$



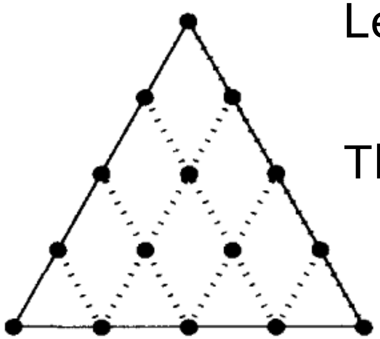
$d = 4$

- Number of nodes in the middle $15 - 12 = 3$

Higher order piecewise polynomials

The a general piecewise polynomial takes the form

$$u = a_1^{(i)} + \qquad (x, y) \in T_i$$
$$+ a_2^{(i)} \; x + a_3^{(i)} y +$$
$$+ a_4^{(i)} \; x^2 + a_5^{(i)} xy + a_6^{(i)} y^2 +$$
$$+ a_7^{(i)} \; x^3 + a_8^{(i)} x^2 y + a_9^{(i)} xy^2 + a_{10}^{(i)} y^3 +$$
$$+ \; a_{11}^{(i)} \; x^4 + a_{12}^{(i)} x^3 y + a_{13}^{(i)} x^2 y^2 + a_{14}^{(i)} xy^3 + a_{14}^{(i)} y^4$$



$d = 4$

Let's suppose I know the solution at the nodes $\{x_j, y_j\}_{j=1}^{12}$:

$$u_j = u(x_j, y_j)$$

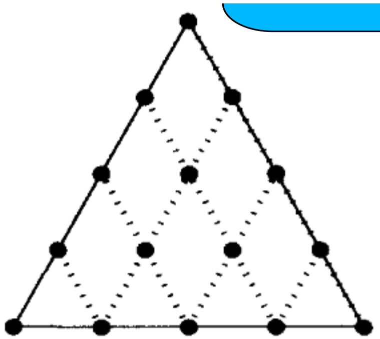
Then the nodes should define uniquely the plane, that is, the values for

$$\left\{a_k^{(i)}\right\}_{j=1}^{12}$$

Higher order piecewise polynomials

The a general piecewise polynomial takes the form

$$\begin{bmatrix} u_1 \\ u_2 \\ \\ \vdots \\ \\ u_{12} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & x_1 & y_1 & x_1^2 & x_1 y_1 & y_1^2 & x_1^3 & \dots & y_1^4 \\ 1 & x_2 & y_2 & x_2^2 & x_2 y_2 & y_2^2 & x_2^3 & \dots & y_2^4 \\ & & & & & & & & \vdots \\ & \vdots & & & \ddots & & & & \\ 1 & x_{12} & \dots & & & & & & y_{12}^4 \end{bmatrix}}_{W} \begin{bmatrix} a_1^{(i)} \\ a_2^{(i)} \\ \\ \vdots \\ \\ a_{12}^{(i)} \end{bmatrix} \qquad (x_j, y_j) \in T_i$$

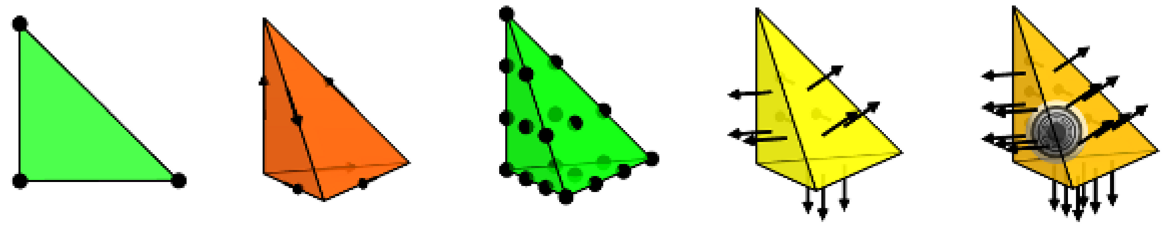


$d = 4$

$\Rightarrow$ W is not singular

Different discretisations of the functional space

Shape functions in C^0 and in C^1 on 1D and or on 2D (quadratic or triangular) elements, the Lagrange polynomials and the Hermite polynomials, number of basis functions, conforming elements… (see lecture, or more in [1], [2], [3])



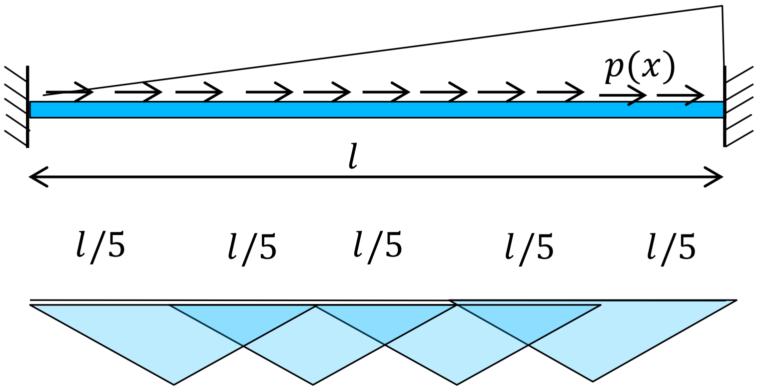
Picture source: http://fenicsproject.org/about/features.html#features

[1]: Brenner&Scott: The Mathematical Theory of FEM, Chapter 3 – Construction of the finite element space

[2]: Zienkiewicz&Taylor: The Finite Elment Method, Chapter 8. ‚Standard' and ‚hierarchical' element shape functions: some general families of C_0 continuity

[3]: Logg&Mardal&Wells: Automated Solution of Differential Equations by the FEM- The Fenics Book, Chapter 3: Common and unusual finite elements

1D Example with linear nodal basis



$p(x) = ax$

Strong form: $-EA\dfrac{d^2u}{dx^2} = p(x)$

$u(0) = u(l) = 0$

Weak form:

$$\int_0^l EA\frac{du}{dx}\frac{d\psi}{dx}dx = \int_0^l p(x)\psi(x)dx$$

Discretisation of the weak form:

$$u(\mathbf{x}) \approx \sum_{i=1}^{4} u_i\,\psi_i(\mathbf{x})$$

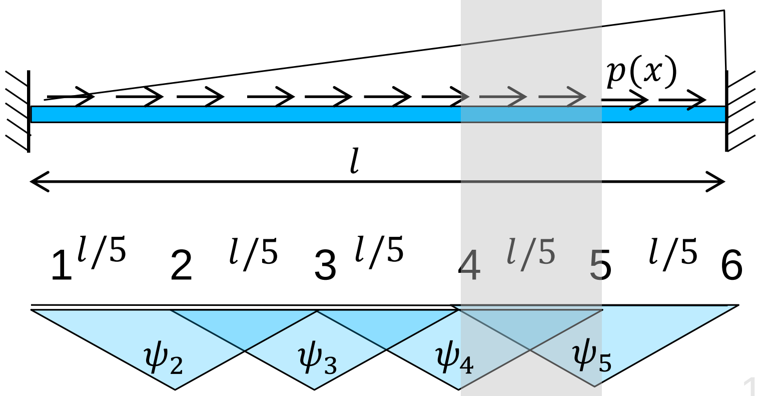
$$\sum_{i=1}^{4} u_j\, EA \underbrace{\int_l \frac{\partial\psi_i(x)}{\partial x}\frac{\partial\psi_j(x)}{\partial x}dx}_{K_{ij}} = \underbrace{\int_l p(x)\psi_j(x)dx}_{f_j}$$

Not efficient to calculate all the elements of the stiffness matrix one by one!

⬇

Calculate element stiffness matrices and assemble

1D Example with linear nodal basis



instead:
Compute stiffness matrix elementwisely and then assemble

$$K_4{}^e(1,1) = EA \int_{\Omega_4} \frac{\partial \psi_4(x)}{\partial x} \frac{\partial \psi_4(x)}{\partial x} dx$$

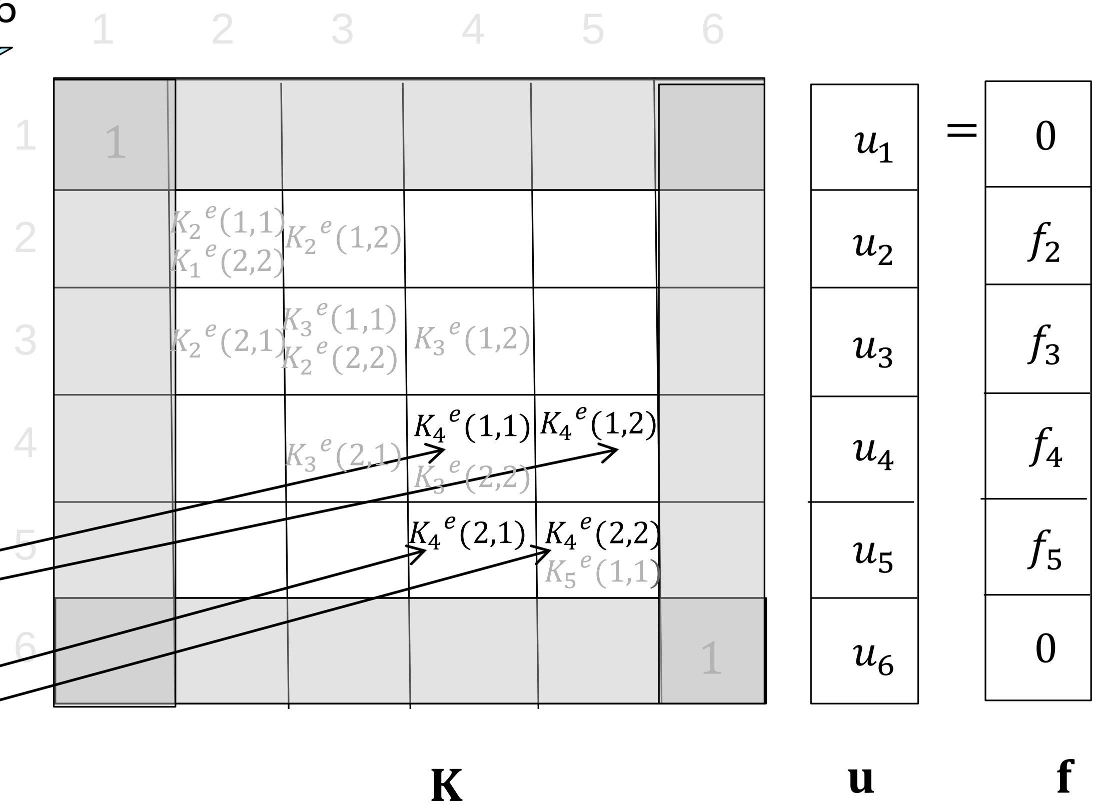
$$K_4{}^e(1,2) = EA \int_{\Omega_4} \frac{\partial \psi_4(x)}{\partial x} \frac{\partial \psi_5(x)}{\partial x} dx$$

$$K_4{}^e(2,1) = EA \int_{\Omega_4} \frac{\partial \psi_5(x)}{\partial x} \frac{\partial \psi_4(x)}{\partial x} dx$$

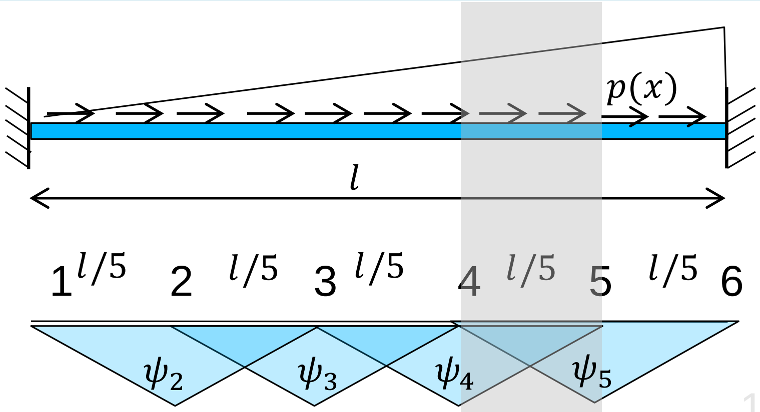
$$K_4{}^e(2,2) = EA \int_{\Omega_4} \frac{\partial \psi_5(x)}{\partial x} \frac{\partial \psi_5(x)}{\partial x} dx$$

$$K_4{}^e = \begin{bmatrix} K_4{}^e(1,1) & K_4{}^e(1,2) \\ K_4{}^e(2,1) & K_4{}^e(2,2) \end{bmatrix}$$

Global stiffness matrix



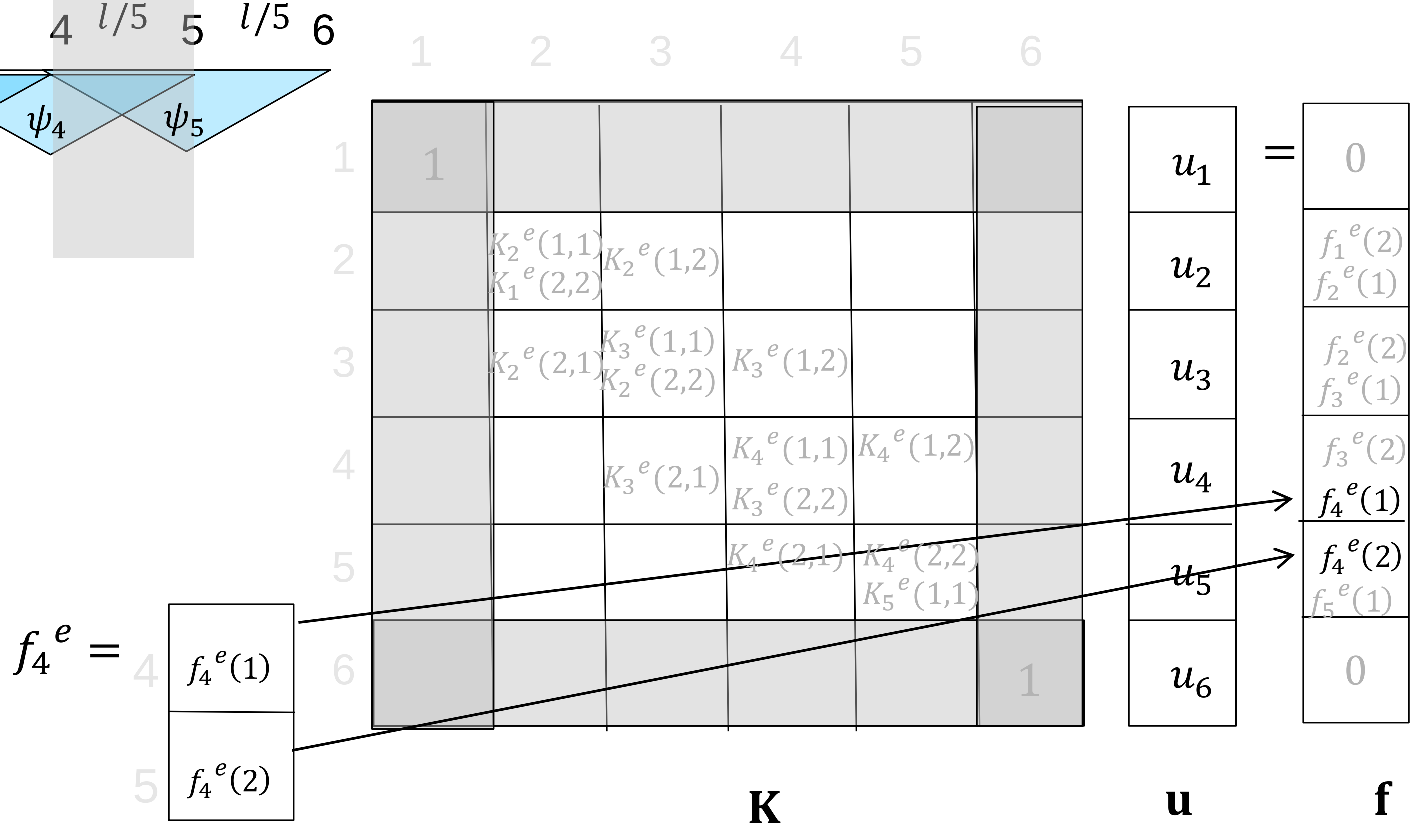
1D Example with linear nodal basis



instead:
Compute stiffness matrix elementwisely and then assemble

$$f_4{}^e(1) = \int_{\Omega_4} p(x)\psi_4(x)dx$$

$$f_4{}^e(2) = \int_{\Omega_4} p(x)\psi_5(x)dx$$



Non-degenerate triangulation

Diameter of a set:

$$\operatorname{diam}(S) = \sup\{\|z_1 - z_2\| \ : \ z_1, z_2 \in S\}$$

Diameter of a triangle:
$\boldsymbol{D_T}$: length of longest side
$\boldsymbol{d_T}$: largest circle contained in T
$\frac{\boldsymbol{d_T}}{\boldsymbol{D_T}}$: measures how skinny the triangle is

Other definitions
T_h:triangulation (set of triangles), with
h: maximal diameter of any triangle in T_h (the length of longest side)

Nondegenerate triangulation:

$$\frac{d_T}{\operatorname{diam}(T)} \geq \rho$$

for all the triangles in the triangulation.

Convergence using piecewise polynomials

$$\text{error} = \|u - u_h\|_E \leq \|u - v\|_E \quad \forall v \in V_h$$

$$\|u - u_h\| \leq \frac{M}{\delta}\|u - v\| \quad \forall v \in V_h$$

Let's compare the best approximation u_h with the proximodel with piecewise linear functions:

$$u_I(x) = \sum_{j=1}^{n} u_{Ij}\Psi_j(x) \quad u_I \in V_h$$

$$\|u - u_h\|_E \leq \|u - u_I\|_E$$

$$\|u - u_h\| \leq \frac{M}{\delta}\|u - u_I\|$$

If I can bound the expression in the r.h.s, I also bound the errors.

Convergence using piecewise linear functions

Theorem:
$\{T_h\}$: non-degenerate family of triangulations of a polygonal domain $\Omega \in R^2$
$u \in H_2$
u_I: piecewise linear approximation
There exists a constant C depending on Ω and the value ρ (see definition of nondegenerate triangulation) such that

$$\|u - u_I\|_{L2} \leq Ch^2 |u|_{H2}$$

$$\|u - u_I\|_{H1} \leq Ch |u|_{H2}$$

where:

$$|u|^2_{H^2(\Omega)} = \int_\Omega \left\{ \left|\frac{\partial^2 u}{\partial x^2}\right|^2 + 2\left|\frac{\partial^2 u}{\partial x \partial y}\right|^2 + \left|\frac{\partial^2 u}{\partial y^2}\right|^2 \right\}$$

(seminorm)

Convergence using piecewise linear functions

$\frac{h}{\sqrt{2}}$	$\|u - u_I\|_{L^2(\Omega)}$	$\|u - u_I\|_{H^1(\Omega)}$
$5.0000 \cdot 10^{-1}$	$5.6484 \cdot 10^{-2}$	$4.1361 \cdot 10^{-1}$
$2.5000 \cdot 10^{-1}$	$1.6022 \cdot 10^{-2}$	$2.2448 \cdot 10^{-1}$
$1.2500 \cdot 10^{-1}$	$4.1305 \cdot 10^{-3}$	$1.1450 \cdot 10^{-1}$
$6.2500 \cdot 10^{-2}$	$1.0405 \cdot 10^{-3}$	$5.7536 \cdot 10^{-2}$

Source: Gockenbach: Understanding and Implementing FEM

Convergence using piecewise higher-order polynomials

Theorem:
$\{T_h\}$: non-degenerate family of triangulations of a polygonal domain $\Omega \in R^2$
$u \in H_{p+1}$
$u_{I,p}$: piecewise d-order approximation
There exists a constant C depending on Ω and the value ρ such that

$$\|u - u_I\|_{L2} \leq Ch^{d+1}|u|_{Hd+1}$$

$$\|u - u_I\|_{H1} \leq Ch^{d}|u|_{Hd+1}$$

where:

$$|u|^2_{H^{d+1}(\Omega)} = \sum_{i+j=d+1} \int_\Omega \left| \frac{\partial^{d+1} u}{\partial x^i \partial y^j} \right|^2$$

See proof in [3]: Brenner&Scott: The Mathematical Theory of FEM, Chapter 4.4

Convergence using piecewise higher-order polynomials

$d = 2$

$\frac{h}{\sqrt{2}}$	$\|u - u_I\|_{L^2(\Omega)}$	$\|u - u_I\|_{H^1(\Omega)}$
$5.0000 \cdot 10^{-1}$	$7.8059 \cdot 10^{-3}$	$1.2655 \cdot 10^{-1}$
$2.5000 \cdot 10^{-1}$	$1.0413 \cdot 10^{-3}$	$3.3340 \cdot 10^{-2}$
$1.2500 \cdot 10^{-1}$	$1.3227 \cdot 10^{-4}$	$8.4444 \cdot 10^{-3}$
$6.2500 \cdot 10^{-2}$	$1.6600 \cdot 10^{-5}$	$2.1180 \cdot 10^{-3}$

$d = 4$

$\frac{h}{\sqrt{2}}$	$\|u - u_I\|_{L^2(\Omega)}$	$\|u - u_I\|_{H^1(\Omega)}$
$5.0000 \cdot 10^{-1}$	$1.1860 \cdot 10^{-4}$	$3.6516 \cdot 10^{-3}$
$2.5000 \cdot 10^{-1}$	$3.8542 \cdot 10^{-6}$	$2.3635 \cdot 10^{-4}$
$1.2500 \cdot 10^{-1}$	$1.2162 \cdot 10^{-7}$	$1.4901 \cdot 10^{-5}$
$6.2500 \cdot 10^{-2}$	$3.8098 \cdot 10^{-9}$	$9.3335 \cdot 10^{-7}$

Source: Gockenbach: Understanding and Implementing FEM

Convergence using piecewise higher-order polynomials

$$\|u - u_I\|_{L^2} \leq Ch^{d+1}|u|_{H^{d+1}}$$

$$\|u - u_I\|_{H^1} \leq Ch^{d}|u|_{H^{d+1}}$$

$$\|u - u_h\|_E \leq \|u - u_I\|_E$$

?

$$\|u - u_h\|_{H^1} \leq \frac{M}{\delta}\|u - u_I\|_{H^1} \leq \frac{M}{\delta} Ch^{d}|u|_{H^{d+1}} = O(h^d)$$

convert to homogeneous problem:

G:known function, $G = g$ on Γ_D

$\hat{u}$:new function that we look for

$$u = G + \hat{u}$$

$$\int_\Omega \kappa(\mathbf{x})\nabla\hat{u}(\mathbf{x}) \cdot \nabla v(\mathbf{x})d\Omega = \int_\Omega f(\mathbf{x})v(\mathbf{x})d\Omega + \int_{\Gamma_N} hv(\mathbf{x})d\Gamma - \int_\Omega \kappa(\mathbf{x})\nabla G(\mathbf{x}) \cdot \nabla v(\mathbf{x})d\Omega$$

from natural/Neumann BC

from essential/Dirichlet BC

Convergence using piecewise higher-order polynomials

First let's suppose homogenous Dirichlet condition:

$$\int_\Omega \kappa(\mathbf{x})\, \nabla u(\mathbf{x}) \cdot \nabla v(\mathbf{x}) d\Omega = \int_\Omega f(\mathbf{x}) v(\mathbf{x}) d\Omega + \int_{\Gamma_N} h v(\mathbf{x}) d\Gamma$$

$$\|u - u_h\|_E \leq \|u - u_I\|_E$$

If

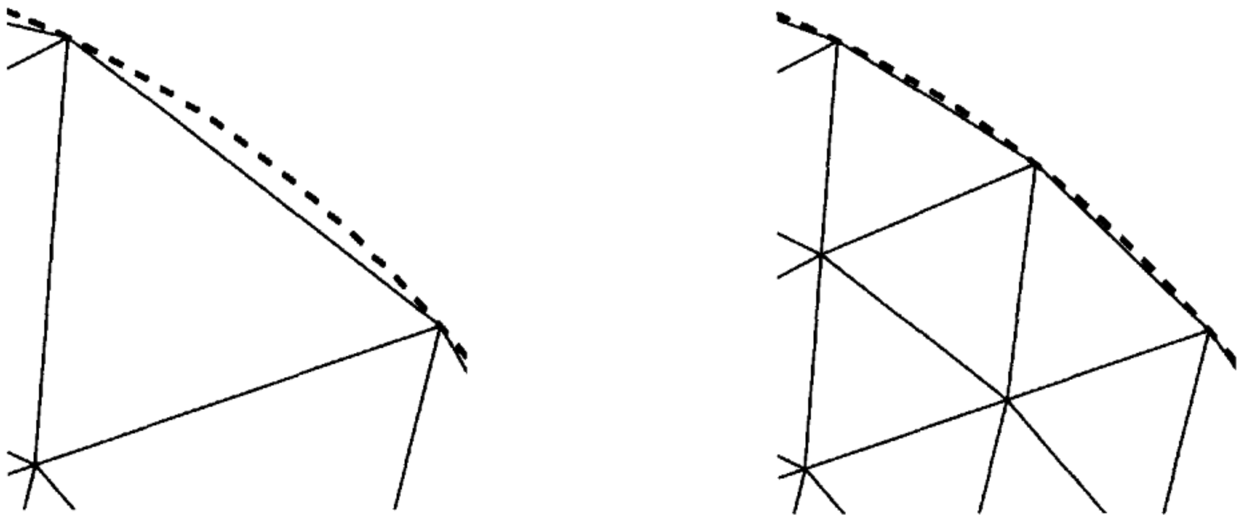
$$0 < k_0 \leq \kappa \leq k_1$$

$$\|u - u_I\|_E = a(u - u_I, u - u_I) \leq M \|u - u_I\|_{H^1} \leq C\, h^d |u|_{H^{d+1}}$$

One can also show for inhomogeneous boundary conditions:

$$\|u - u_h\|_E \leq \sqrt{2} C\, h^d \|u\|_{H^{d+1}}$$ [Chapter 5.3]

Variational crime: curved boundary

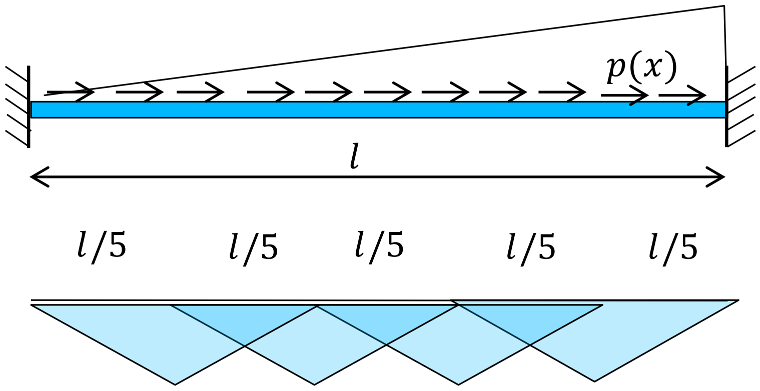


Source: Gockenbach: Understanding and Implementing FEM

Variational crimes ⇒ Céa's lemma and the error estimators may not be valid anymore

but: additional errors can be also estimated (Strang)

Variational crime: numerical integration



$p(x) = ax$

Strong form:

$$-EA\frac{d^2u}{dx^2} = p(x) \qquad u(0) = u(l) = 0$$

Weak form:

$$\int_0^l EA\frac{du}{dx}\frac{d\psi}{dx}\,dx = \int_0^l p(x)\psi(x)\,dx \qquad \forall\psi(x)$$

Discretisation of the weak form:

$$u(\mathbf{x}) \approx \sum_{i=1}^{4} u_i\,\psi_i(\mathbf{x})$$

$$\sum_{i=1}^{4} u_j\,EA\underbrace{\int_l \frac{\partial\psi_i(x)}{\partial x}\frac{\partial\psi_j(x)}{\partial x}dx}_{K_{ij}} = \underbrace{\int_l p(x)\psi_j(x)dx}_{f_j}$$

Example:

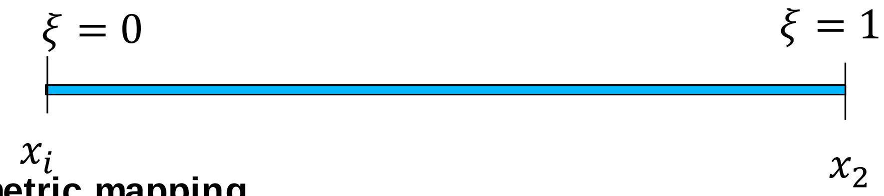
$$f_j = \int_l p(x)\psi_j(x)dx \approx \sum_{k=1}^{n}\omega_k p(x_k)\psi_j(x_k)$$

For more complicated trial functions, p(x) and nonconstant EA(x) difficult to calculate → **Numerical integration** (example: Gauß-quadrature)

Local/ coordinate system, isoparametric mapping 1D

$x \Rightarrow \xi = [0,1]$

local coordinate / global coordinate



coordinate transformation using the **ansatzfunctions** $\Rightarrow$ **isoparametric mapping**

functions of lower order: **subparametric**
functions of higher order: **superparametric**

Shape functions: $N_1(\xi) = 1 - \xi$

$$N_2(\xi) = \xi$$

Transformation from local to global coordinates:

$$x(\xi) = x_i N_1(\xi) + x_{i+1} N_2(\xi) = [N_1(\xi) \quad N_2(\xi)] \begin{bmatrix} x_i \\ x_{i+1} \end{bmatrix} \Rightarrow$$

$$\frac{dx}{d\xi} = x_i \frac{dN_1(\xi)}{d\xi} + x_{i+1} \frac{dN_2(\xi)}{d\xi}$$

$$\frac{dx}{d\xi} = \begin{bmatrix} \frac{dN_1(\xi)}{d\xi} & \frac{dN_2(\xi)}{d\xi} \end{bmatrix} \begin{bmatrix} x_i \\ x_{i+1} \end{bmatrix}$$

Stiffness matrix with isoparametric elements:

$$K_4{}^e(k,l) = EA \int_{\Omega_4} \frac{\partial \psi_i(x)}{x} \frac{\partial \psi_j(x)}{\partial x} dx = EA \int_{\Omega_4} \frac{\partial N_k(\xi)}{\partial \xi} \underbrace{\frac{\partial \xi}{\partial x}}_{\left(\frac{dx}{d\xi}\right)^{-1}} \frac{\partial N_l(\xi)}{\partial \xi} \underbrace{\frac{\partial \xi}{\partial x}}_{\left(\frac{dx}{d\xi}\right)^{-1}} dx$$

$i, j \in [4,5]$

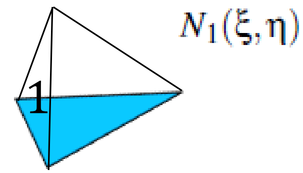
$k, l \in [1,2]$

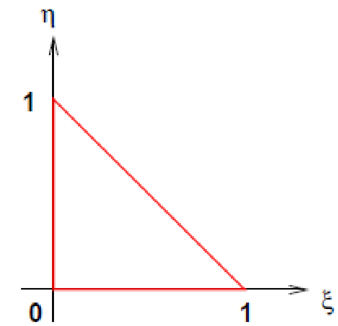
$$K_4{}^e(k,l) = EA \int_0^1 \frac{\partial N_k(\xi)}{\partial \xi} \left(\frac{dx}{d\xi}\right)^{-1} \frac{\partial N_l(\xi)}{\partial \xi} \left(\frac{dx}{d\xi}\right)^{-1} \left|\frac{dx(\xi)}{d\xi}\right| d\xi$$

Local/ coordinate system, isoparametric mapping 2D triangular elements

Basis functions:

$$
\begin{aligned}
N_1(\xi,\eta) &= \xi \\
N_2(\xi,\eta) &= \eta \\
N_3(\xi,\eta) &= 1-\xi-\eta
\end{aligned}
$$





Transformation from local to global coordinates:

$$\begin{pmatrix} x_{glob} \\ y_{glob} \end{pmatrix}(\xi,\eta) = N_1(\xi,\eta)\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + N_2(\xi,\eta)\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} + N_3(\xi,\eta)\begin{pmatrix} x_3 \\ y_3 \end{pmatrix}$$

$$\begin{bmatrix} x_{glob}(\xi,\eta) \\ y_{glob}(\xi,\eta) \end{bmatrix} = \begin{bmatrix} N_1(\xi,\eta) & & N_2(\xi,\eta) & & N_3(\xi,\eta) & \\ & N_1(\xi,\eta) & & N_2(\xi,\eta) & & N_3(\xi,\eta) \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \end{bmatrix}$$

Stiffness matrix:

$$\mathbf{K}_{ij} = \int_{\Omega_{elm}} \begin{pmatrix} \frac{\partial N_j}{\partial x} \\ \frac{\partial N_j}{\partial y} \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{pmatrix} d\Omega_{elm} \qquad i,j \in [1,2,3]$$

Local/ coordinate system, isoparametric mapping 2D triangular elements

Stiffness matrix:

$$\mathbf{K}_{ij} = \int_{\Omega_{elm}} \begin{pmatrix} \frac{\partial N_j}{\partial x} \\ \frac{\partial N_j}{\partial y} \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{pmatrix} d\Omega_{elm} \qquad i,j \in [1,2,3]$$

Stiffness matrix with local coordinates:

$$\boldsymbol{K}_{ij} = \int_0^1 \int_0^{1-\eta} \boldsymbol{J}^{-T} \begin{bmatrix} \frac{\partial N_j}{\partial \xi} \\ \frac{\partial N_j}{\partial \eta} \end{bmatrix} \cdot \boldsymbol{J}^{-T} \begin{bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{bmatrix} |\boldsymbol{J}| d\xi d\eta$$

substitution rule

determinant should not be negative or zero!

$$i,j \in [1,2,3]$$

where:

$$\mathbf{J} = \begin{pmatrix} \frac{\partial x_{glob}}{\partial \xi} & \frac{\partial x_{glob}}{\partial \eta} \\ \frac{\partial y_{glob}}{\partial \xi} & \frac{\partial y_{glob}}{\partial \eta} \end{pmatrix}$$

$$\mathbf{J} = \begin{pmatrix} \sum_{i=1}^{4} \frac{\partial N_i(\xi,\eta)}{\partial \xi} x_i & \sum_{i=1}^{4} \frac{\partial N_i(\xi,\eta)}{\partial \eta} x_i \\ \sum_{i=1}^{4} \frac{\partial N_i(\xi,\eta)}{\partial \xi} y_i & \sum_{i=1}^{4} \frac{\partial N_i(\xi,\eta)}{\partial \eta} y_i \end{pmatrix}$$

$$\frac{\partial N}{\partial \xi} = \frac{\partial N}{\partial x}\frac{\partial x_{glob}}{\partial \xi} + \frac{\partial N}{\partial y}\frac{\partial y_{glob}}{\partial \xi}$$

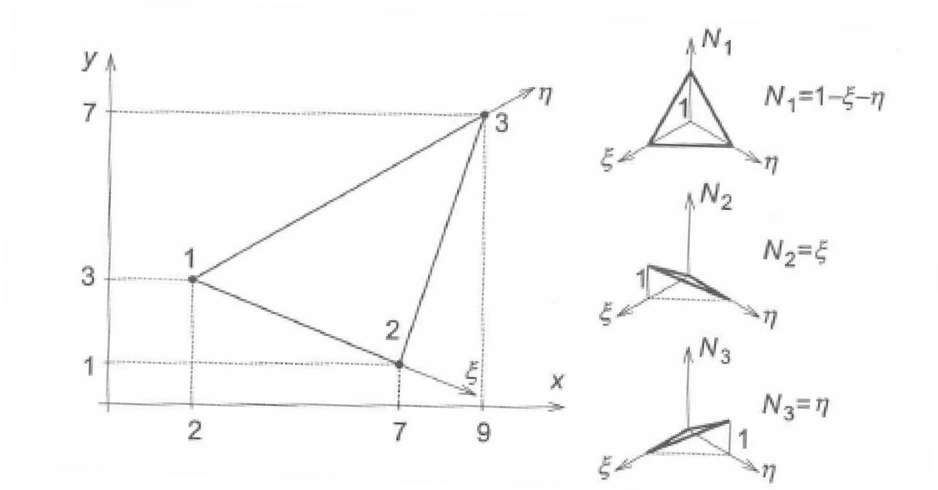
$$\frac{\partial N}{\partial \eta} = \frac{\partial N}{\partial x}\frac{\partial x_{glob}}{\partial \eta} + \frac{\partial N}{\partial y}\frac{\partial y_{glob}}{\partial \eta}$$

$$\Rightarrow \begin{pmatrix} \frac{\partial N}{\partial \xi} \\ \frac{\partial N}{\partial \eta} \end{pmatrix} = \mathbf{J}^T \begin{pmatrix} \frac{\partial N}{\partial x} \\ \frac{\partial N}{\partial y} \end{pmatrix}$$

$$\Downarrow$$

$$\mathbf{J}^{-T} \begin{pmatrix} \frac{\partial N}{\partial \xi} \\ \frac{\partial N}{\partial \eta} \end{pmatrix} = \begin{pmatrix} \frac{\partial N}{\partial x} \\ \frac{\partial N}{\partial y} \end{pmatrix}$$

Local/ coordinate system, isoparametric mapping 2D triangular elements, example

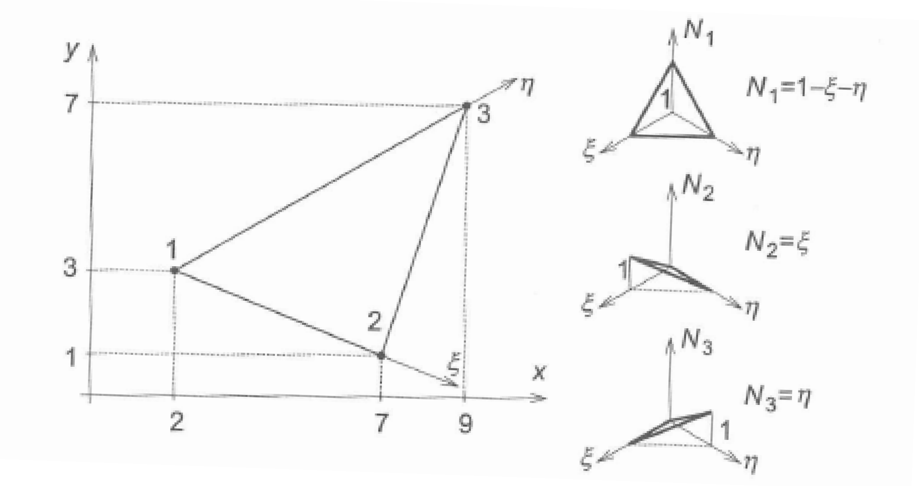


Transformation from local to global coordinates (isoparametric mapping):

$$\begin{bmatrix} x_{glob}(\xi,\eta) \\ y_{glob}(\xi,\eta) \end{bmatrix} = \begin{bmatrix} N_1(\xi,\eta) & & N_2(\xi,\eta) & & N_3(\xi,\eta) & \\ & N_1(\xi,\eta) & & N_2(\xi,\eta) & & N_3(\xi,\eta) \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \end{bmatrix}$$

$$\begin{bmatrix} x(\xi,\eta) \\ y(\xi,\eta) \end{bmatrix} = \begin{bmatrix} (1-\xi-\eta) & & \xi & & \eta & \\ & (1-\xi-\eta) & & \xi & & \eta \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 7 \\ 1 \\ 9 \\ 7 \end{bmatrix}$$

Local/ coordinate system, isoparametric mapping 2D triangular elements, example



Stiffness matrix with local coordinates:

$$\boldsymbol{K}_{ij} = \int_0^1 \int_0^{1-\eta} \boldsymbol{J}^{-\boldsymbol{T}} \begin{bmatrix} \frac{\partial N_j}{\partial \xi} \\ \frac{\partial N_j}{\partial \eta} \end{bmatrix} \cdot \boldsymbol{J}^{-\boldsymbol{T}} \begin{bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{bmatrix} |\boldsymbol{J}| d\xi d\eta \qquad i,j \in [1,2,3]$$

$$\mathbf{J} = \begin{pmatrix} \frac{\partial x_{glob}}{\partial \xi} & \frac{\partial x_{glob}}{\partial \eta} \\ \frac{\partial y_{glob}}{\partial \xi} & \frac{\partial y_{glob}}{\partial \eta} \end{pmatrix}$$

Condition number of the stiffness matrix

Condition number

What happens with the roundoff errors in $\widehat{\mathbf{K}} = \mathbf{LU} = \mathbf{K} + \boldsymbol{\delta}\mathbf{K} \neq \mathbf{K}$

$$\mathbf{K}\mathbf{u} = \mathbf{f} \qquad (\mathbf{K} + \boldsymbol{\delta}K)\widehat{\mathbf{u}} = \mathbf{f} + \boldsymbol{\delta}\mathbf{f} \qquad \frac{\|\widehat{\mathbf{u}} - \mathbf{u}\|}{\|\mathbf{u}\|} \leq \frac{\lambda_{max}}{\lambda_{min}} \frac{\|\boldsymbol{\delta}\mathbf{f}\|}{\|\mathbf{f}\|} \qquad \frac{\lambda_{max}}{\lambda_{min}} = \kappa(\mathbf{K})$$

Condition number of $\mathbf{K}$ with nodal bases with 2D triangular mesh: $O(h^{-2})$

↓

turns il-conditioned for refined mesh!!!

For the Poisson equation

$$\sum_{i=1}^{N} c_i \underbrace{\int_\Omega \nabla\Psi_i(\mathbf{x}) \cdot \nabla\Psi_j(\mathbf{x}) d\Omega}_{K_{ij}} = \int_\Omega f(\mathbf{x})\Psi_j(\mathbf{x}) d\Omega$$