Technische Universität Braunschweig

Numerical methods for PDEs
Adaptive methods, posterior error estimators

Dr. Noemi Friedman

Contents of the course

- Fundamentals of functional analysis

- Abstract formulation FEM
- Spatial (meshing) and functional discretization (the basis functions)
- Convergence, regularity
- Variational crimes
- Numerical integration, implementation

- Mixed formulations (e.g. Stokes)
- Stabilisation for flow problems

- Adaptivity
- Error indicators/estimators

Content of the lecture

Error indicators/estimators

- Introduction
- Different a-posterior estimators / indicators:

  Explicit

  - Quadratic error estimator/indicator
  - Error indicator based on the curvature of the solution
  - Error indicator based on the residual

  Implicit:

  - The element residual error estimator

Introduction

A-priori error estimators:

- give an assymptotic bound (not absolute value) of the erorr, it shows how the error decreases as the mesh is refined
- does not involve the computed solution →‚a priori'

E.g.: $\|u - u_h\|_E \le Ch\,|u_h|_{H2}$, or $\|u - u_h\|_{L2} \le Ch^2 |u_h|_{H2}$

$\|u - u_h\|_E = O(h)$, $\|u - u_h\|_{L2} = O(h^2)$

$\|u - u_h\|_{L\infty} = O(h^2|\log(h)|) \approx O(h^2)$

In 1D: $h = O(N^{-1})$ in 2D: $h = O(N^{-1/2})$ ⟹ $\|u - u_h\|_E = CN^{-1/2}$

A-posteriori error estimators:

- It involves the computed solution and estimates from that the actual error
  - Error indicator: element-wise, it indicates where the error is larger
  - Error estimator: it shows when the probel has been solved accurately enough → the algorithm can be halted

1. Quadratic error estimator (explicit) / indicator (but not efficient)

Solve the probelm with piecewise linear approximation and estimate the true solution by the solution from quadratic approximation:

$$\|u - u_h\|_E \approx \left\| u_h^{(2)} - u_h \right\|_E$$

Where:

- u: the true solution
- u_h: solution by piece-wise **linear** approximation
- $u_h^{(2)}$: solution by piece-wise **quadratic** approximation

Computationally very expensive!!

2. Error estimator based on the curvature (source : [Gockenbach])

For the rest of the estimators let's suppose the BVP:

$$
\begin{aligned}
-\nabla \cdot (\kappa \nabla u) &= f \text{ in } \Omega, \\
u &= g \text{ on } \Gamma_1, \\
\kappa \frac{\partial u}{\partial n} &= h \text{ on } \Gamma_2
\end{aligned}
$$

Let's start from the a-priori bound of the L^∞ norm.
The L^∞ norm (the essential supremum) of the error is bounded by:

$$\|u - u_h\|_{L^\infty(\Omega)} \leq C h^2 \left|\log(h)\right| \|u\|_{W^{2,\infty}(\Omega)}$$

In an element-wise manner it can be expressed as:

$$\|u - u_h\|_{L^\infty(\Omega)} \leq C \left|\log(h)\right| \max_{T \in \mathcal{T}_h} h_T^2 |u|_{W^{2,\infty}(T)}$$

2. Error indicator based on the curvature (source : [Gockenbach])

In an element-wise manner it can be expressed as:

$$\|u - u_h\|_{L^\infty(\Omega)} \leq C\,|\log(h)| \max_{T \in \mathcal{T}_h} h_T^2 |u|_{W^{2,\infty}(T)}$$

we do not need

this is what we need to estimate

When an estimator is needed, then the constant C has to be somehow approximated. Here we need only an error indicator, we do not really care about the value of the constant. As $|\log(h)|$ is changing much slower then h^2, we can include it in the constant C.

$W^{k,\infty} = \{u : \Omega \to \mathbf{R} \mid u$ and its partial derivatives up to order k are in $L^\infty(\Omega)\}$

By definition, $|u|_{W^{2,\infty}(T)}$ is the largest of

$$\max\left\{\left|\frac{\partial^2 u}{\partial x^2}(x,y)\right| : (x,y) \in T\right\},$$

$$\max\left\{\left|\frac{\partial^2 u}{\partial y^2}(x,y)\right| : (x,y) \in T\right\},$$

$$\max\left\{\left|\frac{\partial^2 u}{\partial x \partial y}(x,y)\right| : (x,y) \in T\right\}.$$

2. Error indicator based on the curvature (source : [Gockenbach])

The term:

$$\max\left\{\left|\frac{\partial^2 u}{\partial x^2}(x,y)\right| : (x,y)\in T\right\},$$

$$\max\left\{\left|\frac{\partial^2 u}{\partial y^2}(x,y)\right| : (x,y)\in T\right\},$$

$$\max\left\{\left|\frac{\partial^2 u}{\partial x \partial y}(x,y)\right| : (x,y)\in T\right\}.$$

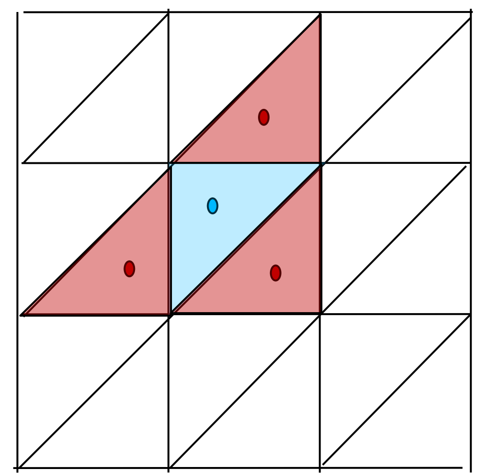
can be estimated by the largest of [Erikson-Johnson]:

at the center of the triangle

neighboring triangle

$$\frac{\left|\frac{\partial u_h}{\partial x}(x_T, y_T) - \frac{\partial u_h}{\partial x}(x_{\hat{T}}, y_{\hat{T}})\right|}{\left\|(x_T, y_T) - (x_{\hat{T}}, y_{\hat{T}})\right\|},$$

$$\frac{\left|\frac{\partial u_h}{\partial y}(x_T, y_T) - \frac{\partial u_h}{\partial y}(x_{\hat{T}}, y_{\hat{T}})\right|}{\left\|(x_T, y_T) - (x_{\hat{T}}, y_{\hat{T}})\right\|},$$



3. Error indicator based on the residual (source : [Gockenbach])

The weak formulation: $a(u,v) = l(v) \quad \forall v \in V$

Galerkin orthogonality: $a(u-u_h,v) = 0 \quad \forall v \in V_h \subset V$

$$a(u-u_h,v) = a(u,v) - a(u_h,v) = l(v) - a(u_h,v) \; \forall v \in V$$
$$= 0 \qquad \forall v \in V_\mathrm{h}$$

For the used BVP with homogenous BC:

$$a(u,v) = \int_\Omega \kappa \nabla u \cdot \nabla v,$$
$$\ell(v) = \int_\Omega f v + \int_{\Gamma_2} h v.$$

We start with:

$$a(u-u_h,v) = l(v) - a(u_h,v)$$

3. Error indicator based on the residual (source : [Gockenbach])

$$a(u-u_h,v) = l(v) - a(u_h,v)$$

$$a(u,v) = \int_\Omega \kappa \nabla u \cdot \nabla v,$$

$$\ell(v) = \int_\Omega f v + \int_{\Gamma_2} h v.$$

$$a(u-u_h,v) =$$

$$= \sum_{T\in\mathcal{T}_h} \left\{ \int_T f v - \int_T \kappa \nabla u_h \cdot \nabla v + \int_{\partial T \cap \Gamma_2} h v \right\}$$

Green's identity

$$= \sum_{T\in\mathcal{T}_h} \left\{ \int_T f v + \int_T \nabla \cdot (\kappa \nabla u_h)\, v - \int_{\partial T} \kappa \frac{\partial u_h}{\partial n} v + \int_{\partial T \cap \Gamma_2} h v \right\}$$

As the term is written elementwisely, u_h within the triangle is smooth

$$= \sum_{T\in\mathcal{T}_h} \left\{ \int_T (f + \nabla \cdot (\kappa \nabla u_h))\, v + \int_{\partial T \cap \Gamma_2} \left(h - \kappa \frac{\partial u_h}{\partial n}\right) v - \int_{\partial T \setminus \Gamma_2} \kappa \frac{\partial u_h}{\partial n} v \right\}$$

$$= \sum_{T\in\mathcal{T}_h} \left\{ \int_T \underbrace{(f + \nabla \cdot (\kappa \nabla u_h))}_{=r}\, v + \int_{\partial T \cap \Gamma_2} \underbrace{\left(h - \kappa \frac{\partial u_h}{\partial n}\right)}_{=R} v - \int_{\partial T \setminus \Gamma_2} \kappa \frac{\partial u_h}{\partial n} v \right\}$$

$= r$ (element by element residual in the interior)

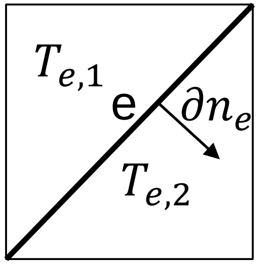
$= R$ (element by element residual on the Neumann B.)

3. Error indicator based on the residual (source : [Gockenbach])

$$a(e_h, v) = \sum_{T\in\mathcal{T}_h}\left\{\int_T (f + \nabla\cdot(\kappa\nabla u_h))\,v + \int_{\partial T\cap\Gamma_2}\left(h - \kappa\frac{\partial u_h}{\partial n}\right)v - \int_{\partial T\setminus\Gamma_2}\kappa\frac{\partial u_h}{\partial n}v\right\}$$

$= r$ (element by element residual in the interior)

$= R$ (element by element residual on the Neumann B.)



$$\sum_{T\in\mathcal{T}_h}\int_{\partial T\setminus\Gamma_2}\kappa\frac{\partial u_h}{\partial n}v = \sum_{e\in\mathcal{I}_h}\left\{\int_e \kappa\frac{\partial u_h|_{T_{e,1}}}{\partial n_e}v - \int_e \kappa\frac{\partial u_h|_{T_{e,2}}}{\partial n_e}v\right\}$$

$$= \sum_{e\in\mathcal{I}_h}\int_e\left(\kappa\frac{\partial u_h|_{T_{e,1}}}{\partial n_e} - \kappa\frac{\partial u_h|_{T_{e,2}}}{\partial n_e}\right)v$$

$$= \sum_{e\in\mathcal{I}_h}\int_e\left[\kappa\frac{\partial u_h}{\partial n_e}\right]_e v,$$

for all the inner edges (on the D.B. it is zero anyway)

with $[f]_e$ the jump of f across the e edge

$$a(e_h, v) = \sum_{T\in\mathcal{T}_h}\left\{\int_T (f + \nabla\cdot(\kappa\nabla u_h))\,v + \int_{\partial T\cap\Gamma_2}\left(h - \kappa\frac{\partial u_h}{\partial n}\right)v - \sum_{e\in\mathcal{I}_h}\int_e\left[\kappa\frac{\partial u_h}{\partial n_e}\right]_e v\right.$$

3. Error indicator based on the residual (source : [Gockenbach])

$$a(e_h, v) = \sum_{T\in\mathcal{T}_h}\left\{\int_T (f + \nabla\cdot(\kappa\nabla u_h))\, v + \int_{\partial T\cap\Gamma_2}\left(h - \kappa\frac{\partial u_h}{\partial n}\right) v - \sum_{e\in\mathcal{I}_h}\int_e \left[\kappa\frac{\partial u_h}{\partial n_e}\right]_e v\right.$$

$= r$ (element by element residual in the interior)

$$a(e_h, v) = \sum_{T\in\mathcal{T}_h}\int_T r v + \sum_{e\in\mathcal{E}_h}\int_e R v.$$

for all edges

with

$$R|_e = \begin{cases} 0, & e\subset\Gamma_1, \\ h - \kappa\frac{\partial u_h}{\partial n}, & e\subset\Gamma_2, \\ \left[\kappa\frac{\partial u_h}{\partial n_e}\right]_e, & \text{otherwise} \end{cases}$$

3. Error indicator based on the residual (source : [Gockenbach])

$$a(e_h, v) = \sum_{T \in \mathcal{T}_h} \int_T r v + \sum_{e \in \mathcal{E}_h} \int_e R v.$$

v_h: piecewise linear

$$a(e_h, v - v_h) = a(e_h, v) - a(e_h, v_h)$$

$= 0$ as $v_h \in V_h$ (Galerkin orthogonality)

So we can replace $a(e_h, v)$ with $a(e_h, v - v_h)$

$$a(e_h, v) = \sum_{T \in \mathcal{T}_h} \int_T r(v - \overline{v}_h) + \sum_{e \in \mathcal{E}_h} \int_e R(v - \overline{v}_h)$$

(Cauchy-Schwarz ineq.)

$$\leq \sum_{T \in \mathcal{T}_h} \|r\|_{L^2(T)} \|v - \overline{v}_h\|_{L^2(T)} + \sum_{e \in \mathcal{E}_h} \|R\|_{L^2(e)} \|v - \overline{v}_h\|_{L^2(e)}$$

$$\|v - \overline{v}_h\|_{L^2(T)} \leq C h_T \|v\|_{H^1(\tilde{T})},$$

$$\|v - \overline{v}_h\|_{L^2(e)} \leq C h_T^{1/2} \|v\|_{H^1(\tilde{T})},$$

for $v \in H^1$

Theorem Ainsworth and Oden

3. Error indicator based on the residual (source : [Gockenbach])

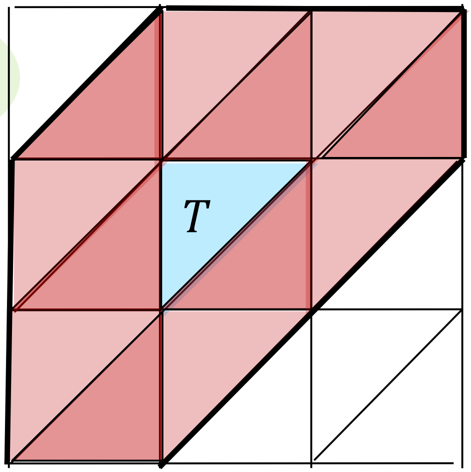
$$a(e_h, v) \le \sum_{T\in\mathcal{T}_h} \|r\|_{L^2(T)} \|v-\overline{v}_h\|_{L^2(T)} + \sum_{e\in\mathcal{E}_h} \|R\|_{L^2(e)} \|v-\overline{v}_h\|_{L^2(e)}$$

$$\|v-\overline{v}_h\|_{L^2(T)} \le C h_T \|v\|_{H^1(\tilde{T})},$$

$$\|v-\overline{v}_h\|_{L^2(e)} \le C h_T^{1/2} \|v\|_{H^1(\tilde{T})},$$

for $v \in H^1$

any edge of T



$\tilde{T}$

patch around T: $\tilde{T} = \{T_1 \in \mathcal{T}_h \; : \; T_1 \cap T \neq \emptyset\}$

$$a(e_h, v) \le C \sum_{T\in\mathcal{T}_h} h_T \|r\|_{L^2(T)} \|v\|_{H^1(\tilde{T})} + C \sum_{e\in\mathcal{E}_h} h_T^{1/2} \|R\|_{L^2(e)} \|v\|_{H^1(\tilde{T})}$$

(Cauchy-Schwarz ineq.)

$$\le C \left\{ \sum_{T\in\mathcal{T}_h} h_T^2 \|r\|^2_{L^2(T)} + \sum_{e\in\mathcal{E}_h} h_T \|R\|^2_{L^2(e)} \right\}^{1/2} \times \left\{ \sum_{T\in\mathcal{T}_h} \|v\|^2_{H^1(\tilde{T})} + \sum_{e\in\mathcal{E}_h} \|v\|^2_{H^1(\tilde{T}_e)} \right\}^{1/2}$$

3. Error indicator based on the residual (source : [Gockenbach])

$$a(e_h, v) \leq C \left\{ \sum_{T \in \mathcal{T}_h} h_T^2 \|r\|_{L^2(T)}^2 + \sum_{e \in \mathcal{E}_h} h_T \|R\|_{L^2(e)}^2 \right\}^{1/2} \times \left\{ \sum_{T \in \mathcal{T}_h} \|v\|_{H^1(\tilde{T})}^2 + \sum_{e \in \mathcal{E}_h} \|v\|_{H^1(\tilde{T}_e)}^2 \right\}^{1/2}$$

$$\sum_{T \in \mathcal{T}_h} \|v\|_{H^1(\tilde{T})}^2 + \sum_{e \in \mathcal{E}_h} \|v\|_{H^1(\tilde{T}_e)}^2 \leq C \sum_{T \in \mathcal{T}_h} \|v\|_{H^1(T)}^2 = C \|v\|_{H^1(\Omega)}^2$$

$$\int_\Omega \kappa \nabla e_h \cdot \nabla v \leq C \|v\|_{H^1(\Omega)} \left\{ \sum_{T \in \mathcal{T}_h} h_T^2 \|r\|_{L^2(T)}^2 + \sum_{e \in \mathcal{E}_h} h_T \|R\|_{L^2(e)}^2 \right\}^{1/2}$$

3. Error indicator based on the residual (source : [Gockenbach])

$$\int_\Omega \kappa \nabla e_h \cdot \nabla v \le C \|v\|_{H^1(\Omega)} \left\{ \sum_{T \in \mathcal{T}_h} h_T^2 \|r\|_{L^2(T)}^2 + \sum_{e \in \mathcal{E}_h} h_T \|R\|_{L^2(e)}^2 \right\}^{1/2}$$

Taking $v = e_h$ and using the V-ellipticity of $a(\cdot,\cdot)$ yields

$$\|e_h\|_{H^1(\Omega)} \le C \left\{ \sum_{T \in \mathcal{T}_h} h_T^2 \|r\|_{L^2(T)}^2 + \sum_{e \in \mathcal{E}_h} h_T \|R\|_{L^2(e)}^2 \right\}^{1/2},$$

or, regrouping the boundary integrals and noting that most edges belong to two triangles,

$$\|e_h\|_{H^1(\Omega)} \le C \left\{ \sum_{T \in \mathcal{T}_h} \left(h_T^2 \|r\|_{L^2(T)}^2 + \frac{1}{2} h_T \|R\|_{L^2(\partial T)}^2 \right) \right\}^{1/2}.$$

Based on this indicator, the *explicit residual* indicator is defined by

$$\boxed{\eta_T = \left\{ h_T^2 \|r\|_{L^2(T)}^2 + \frac{1}{2} h_T \|R\|_{L^2(\partial T)}^2 \right\}^{1/2}}$$

4. Element residual error estimator (implicit) (source: [Bank and Weiser] via [Gockenbach])

PDE satisfied by the error ($e_h = u - u_h$):

$$-\nabla \cdot (\kappa \nabla e_h) = -\nabla \cdot (\kappa \nabla u) + \nabla \cdot (\kappa \nabla u_h) = f + \nabla \cdot (\kappa \nabla u_h)$$

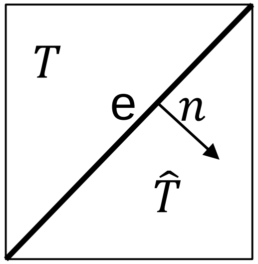
Valid in the interrior of each triangle, where u_h is smooth:
We need to define BC to be able to solve the PDE:

$$-\nabla \cdot (\kappa \nabla e_h) = f + \nabla \cdot (\kappa \nabla u_h) \text{ in } T,$$

$$\kappa \frac{\partial e_h}{\partial n} = \kappa \frac{\partial u}{\partial n} - \kappa \frac{\partial u_h}{\partial n} \text{ on } \partial T.$$

$e_h = u - u_h$

not known
but let's estimate
It!



On the inner edges $e \in \partial T$:

$$\frac{\partial u}{\partial n} = \frac{1}{2}\left(\nabla u_h|_T \cdot n + \nabla u_h|_{\hat{T}} \cdot n\right)$$

$$\left\langle \kappa \frac{\partial u_h}{\partial n} \right\rangle = \frac{1}{2}\left(\kappa \nabla u_h|_T \cdot n + \kappa \nabla u_h|_{\hat{T}} \cdot n\right)$$

4. Element residual error estimator (implicit) (source: [Bank and Weiser] via [Gockenbach])

On the Neumann Boundary:

$$\kappa \frac{\partial e_h}{\partial n} = h - \kappa \frac{\partial u_h}{\partial n}$$

The error is estimated by solving the PDE:

$$-\nabla \cdot (\kappa \nabla e_h) = f + \nabla \cdot (\kappa \nabla u_h) \text{ in } T$$

$$\kappa \frac{\partial e_h}{\partial n} = \left\langle \kappa \frac{\partial u_h}{\partial n} \right\rangle - \kappa \frac{\partial u_h}{\partial n} \text{ if } e \in \mathcal{I}_h$$

$$\kappa \frac{\partial e_h}{\partial n} = h - \kappa \frac{\partial u_h}{\partial n} \text{ if } e \subset \Gamma_2,$$

$$e_h = 0 \text{ if } e \subset \Gamma_1.$$

The weak form of the PDE:

$$\int_T \kappa \nabla e_h \cdot \nabla v = \int_T f v - \int_T \kappa \nabla u_h \cdot \nabla v + \int_{\partial T} \left\langle \kappa \frac{\partial u_h}{\partial n} \right\rangle v \text{ for all } v \in V_T$$

4. Element residual error estimator (implicit) (source: [Bank and Weiser] via [Gockenbach])

The weak form of the PDE:

$$\int_T \kappa \nabla e_h \cdot \nabla v = \int_T f v - \int_T \kappa \nabla u_h \cdot \nabla v + \int_{\partial T} \left\langle \kappa \frac{\partial u_h}{\partial n} \right\rangle v \text{ for all } v \in V_T$$

with:

$$V_T = \left\{ v \in H^1(T) \, : \, v = 0 \text{ on } \partial T \cap \Gamma_1 \right\}$$

From the solution, e_h, of the PDE the element residual error estimate is:

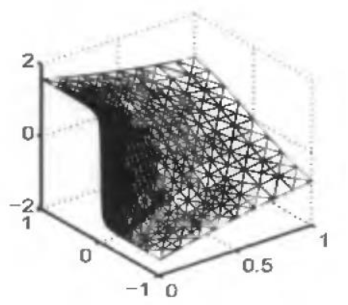
$$\|e_h\|_{E,T} = \left[\int_T \kappa \nabla e_h \cdot \nabla e_h \right]^{1/2}$$

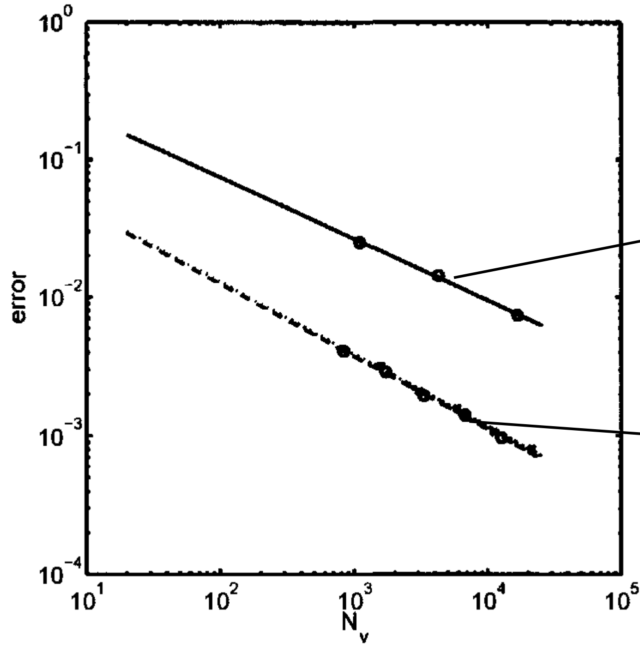
In the inner elements we have pure Neumann BC, solution exists only when the compatibility condition is satisfied. It was shown that the PDE can be solved on some special approximating subspace (quadratic piece-wise polyn, supposin zero at the vertices, sometimes extended by one additional cubic shape function – bubble func.).

$$\bar{e}_h \in \mathcal{M}(T), \; \int_T \kappa \nabla \bar{e}_h \cdot \nabla v = \int_T f v - \int_T \kappa \nabla u_h \cdot \nabla v + \int_{\partial T} \left\langle \kappa \frac{\partial u_h}{\partial n} \right\rangle v \text{ for all } v \in \mathcal{M}(T)$$

4. Element residual error estimator (implicit) (source: [Bank and Weiser] via [Gockenbach])

Example:





Uniform refinement

Adaptive scheme with element residual error indicator

Three point estimator: $\|u - u_h\|_E \doteq 0.141 \cdot N^{-0.526}$

Four point estimator: $\|u - u_h\|_E \doteq 0.144 \cdot N^{-0.523}$