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2017-18

# Introduction to PDEs and Numerical Methods (PDEs 1): Preperation for the test

**Exercise 1:** *Definition and/or basic understanding of the terms:*

- divergence, gradient, curl, nabla operator
- inner product, norms (of a vector/matrix and of a function)
- eigenvalue/eigenvector
- classification of PDEs
- wave equation, Laplace equation, heat equation
- separation of variables, separation-ansatz, separation constant
- well-posedness
- Projection theory, Grammian
- FD method
- computational stencil
- semi- discrete approximation of the heat equation
- Euler forward/Euler backward/Theta methods (forward/backward/centered differences)
- unconditionally stable
- solving system of equations in the basis defined by the eigenvectors
- convergency, stability, consistency, connection in between
- consistency order, convergence order
- Von Neumann stability analysis/gain factor
- connection between the gain factor and stability/positivity/oscillations
- different FD schemes of the heat equation (1D/2D) and their stability criteria
- direct/iterative solvers, choice of solvers

- steepest descent method, conjugate gradient method
- turning a linear system of equations to a minimization problem, conditions of doing so
- positive-definiteness
- Dirichlet/Neumann boundary conditions
- weighted residual methods (principle, examples)
- week/variational form of a boundary value problem
- divergence theorem, Green's identity
- FEM, Bubnov Galerkin/Petrov Galerkin methods
- pointwise collocation, subdomain collocation
- test/weighting function, ansatz/basis function
- nodal/Lagrange basis, isoparametric elements, isoparametric mapping
- characteristic function, Dirac-delta function
- stiffness matrix
- Gauß quadrature, numerical integration

**Exercise 2:** *Classification and analytical solution of PDEs*

- Tell whether a given defferential operator is linear or not
- Classify PDEs (stationary/instationary, order of the PDE, linearity, hyperbolic/elliptic/parabolic)
- Give examples of hyperbolic/elliptic/parabolic PDEs
- Prove whether a given function is a solution of a given PDE (either with given boundary condition or if not, find the boundary conditions)
- Define a certain parameter/function in a given PDE to assure that a given function is a solution of that PDE with predefined boundary/initial conditions
- Solve simple ODEs analyticaly
- Solve simple PDEs analyticaly
- Write the Fourier-series of a given function
- Apply the differential operators (grad/div/curl)

**Exercise 3:** *FD method*

- Descretize a given PDE using FD method (with given discretization method(s)). Give resulted equation for a typical element/timestep, write a script that solves the system.
- Derive a certain finite difference scheme for a given problem using given stencils (for example derive an approximation of $u''(x)$ using the values $u(x-h)$, $u(x)$ and $u(x+h)$). Determine the truncation error, order of the method, error constant.
- Define whether a given scheme for a given problem is cosistent or not.
- Define from a given matrix form/ MATLAB code, what scheme is used
- Define whether a given scheme is stable or not with eigenvalue analysis. Give stability condition when not unconditionaly stable.
- Use the von-Neumann stability analysis to define whether a given scheme gives stable/positive/oscillatory solution. Define the condition of stability/positivity/no oscillation if these properties are not unconditionaly satisfied.

**Exercise 4:** *Krylov subspace methods as (Petrov-)Galerkin approximations to linear systems*

- Connection of minimisation of quadratic functional to a solution of linear system. Discrete weak formulation as an optimality condition. Assumptions on the matrix (positive definitness, symmetry) and why those assumptions are important.
- Krylov subspace. Definition of conjugate gradients (CG) as a minimisation over a Krylov subspace, connection to Galerkin approximation. Derivation of an optimality condition for conjugate gradients.
- Gramm-Schmidt orthogonalisation and A-orthogonalisation. Why this orthogonalisation is important for CG.
- Definition of GMRES method as a minimisation of residual function over a Krylov subspace. Derivation of optimality conditions.

**Exercise 5:** *Weighted residual methods*

- Derive the variational/week form of a given PDE (in 1D or in higher dimension) and the vector spaces of the solution and of the weighting functions
- Calculate the element stiffness matrix $K^{(e)}$ for a given set of ansatz functions with a given bilinear form
- Derive from the weak form, by Bubnov-Galerkin method, the linear system of equations, that needs to be solved (using given ansatz/weighting functions)
- Calculate the right hand side term of the week form $l$ of a given boundary value problem with homogeneous/inhomogeneous Neumann/Dirichlet boundary condition(s) for a given set of ansatz functions, and a given linear form. (Use numerical integration when needed).
- Numerical integration in 1D for arbitrary function with quadrature rule, define the accuracy of the integration rule, or determine the needed number of points when $f(x)$ is polynomial of given order, to get exact result with the numerical integration.

- Assemble the global stiffness matrix from the element stiffness matrices for a given problem (piece-wise linear ansatz functions)
- Apply different boundary conditions
- Solve a small system with FEM and plot its solution
- Define linear ansatzfunctions for a given element (line/triangular/quadrilateral)
- Define isoparametric mapping function from a masterelement onto an element with a pre-defined geometry located in a given coordinate system, with a given set of ansatzfunctions
- Convert gradient or integral defined on a global coordinate system to an expression defined on local coordinate system from a given mapping from local to global coordinates.
- Sketch ansatzfunctions for a given (line/triangular/quadrilateral) element