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Numerical Methods for PDEs (PDEs 2): Convergence, basis functions, FEM implementation and adaptivity

Exercise 1: *FEM: Triangulation properties, sparsity of the stiffness matrix* **(6 points)**

(a) Let d_T be the diameter of the largest circle contained in a triangle T. Prove that

$$\frac{d_T}{\mathrm{diam}(T)} \leq \frac{1}{\sqrt{3}}$$

(3 points)

(b) What is the role of this measure in the convergence analysis? (1 points)

(c) A mesh of quadratic Lagrange triangles has two types of basis functions, those corresponding to vertex nodes and those corresponding to midpoint nodes. How many nodes are adjacent to a typical vertex node? To a typical midpoint node? How many nonzeros lie in each row of the corresponding stiffness matrix? (2 points)

Exercise 2: *A priori error estimates* **(4 points)**

Suppose $\{T_h\}$ is a nondegenerate family of triangulations of a polygonal domain $\Omega \in \Re^2$ and suppose $f \in H^{d+1}(\Omega)$.

(a) If $f_I \in P_h^d$ is the piecewise polynomial interpolant of degree d of f, what can we know about the bound of the L^2 norm of the interpolation error. Please explain every expression in the bound. (2 points)

(b) This error bound is used to show convergence of the Galerkin method. What is the connection between the interpolation error and the error of the Galerkin approximation? (2 points)

Exercise 3: *C_1 basis functions* **(7 points)**

Let us define piece-wise Hermite polynomials, to span the space of $C^1(\Omega)$ piece-wise polynomials of total degree d.

Suppose $\Omega \subset R$, so a 1D domain. Define the local basis functions defined over one element in its local coordinate system $\Phi_i(\xi) : \Omega_e \to \mathbb{R}, \quad \Omega_e = [-1, 1]$ as the linear combination of the monomials:

$$\phi_i(\xi) = a_{i0} + a_{i1}\xi + a_{i1}\xi^2 + \ldots + a_{id}\xi^d.$$

The vector of coefficients

$$a_i^T = \begin{bmatrix} a_{i0} & a_{i1} & \ldots & a_{id} \end{bmatrix}$$

can be calculated from i linear system of equations:

$$\tilde{B}a_i = b_i.$$

What is the matrix $\tilde{B}$ and what are the right hand sides b_i when a two-node element is used? The two nodes are at the coordinates $\xi_1 = -1$ and $\xi_2 = 1$. Draw a draft of all the basis functions defined over the element.

Hint:

- Use $N = 4$ and $d = 3$ to have a well defined system of equations.

Draft of basis functions $\{\Phi_i\}_{i=1}^N$:

Determination of matrix $\tilde{B}$:

And the right hand sides $\{b_i\}_{i=1}^N$:

Exercise 4: C_0 *basis functions* **(7 points)**

Now we define piece-wise Lagrange polynomials to span the space of all $C^0(\Omega)$ piece-wise polynomials of total degree d. Let $\Omega = (0,1) \times (0,1)$ be a unit square and consider a uniform triangulation of Ω created by dividing Ω into n^2 sub-squares, each with side length $h = 1/n$, and then dividing each sub-square into two triangles. Consider two different triangulations:

1. $\{T_1\}$: uniform linear Lagrange triangulation with $n = 2k$ (with $2(2k)^2$ triangles, $d = 1$);
2. $\{T_2\}$: uniform quadratic Lagrange triangulation with $n = k$ (with $2k^2$ triangles, $d = 2$);

- How many nodes are needed per inner edges in the triangulations $\{T_1\}$ and $\{T_2\}$ and why?
- Which triangulation has more nodes (you can try for example $k = 4$, and draw a draft of the mesh)?
- Which triangulation has more nodes (you can try for example $k = 4$, and draw a draft of the mesh)?
- Which resulting stiffness matrix: $K_{ij} = \int_\Omega \nabla\phi_i \cdot \nabla\phi_j$ has more nonzero elements (which one is more sparse) and why? What is the maximum number of nonzero elements per row for the two triangulations?
- Draw one triangle and its nodes that has to be used when $d = 4$. Explain the number of the nodes.

number of points, n	Points, x_i	Weights, w_i
1	0	2
2	$\pm\sqrt{\frac{1}{3}}$	1
3	0	$\frac{8}{9}$
	$\pm\sqrt{\frac{3}{5}}$	$\frac{5}{9}$
4	$\pm\sqrt{\frac{3}{7}-\frac{2}{7}\sqrt{\frac{6}{5}}}$	$\frac{18+\sqrt{30}}{36}$
	$\pm\sqrt{\frac{3}{7}+\frac{2}{7}\sqrt{\frac{6}{5}}}$	$\frac{18-\sqrt{30}}{36}$
5	0	$\frac{128}{225}$
	$\pm\frac{1}{3}\sqrt{5-2\sqrt{\frac{10}{7}}}$	$\frac{322+13\sqrt{70}}{900}$
	$\pm\frac{1}{3}\sqrt{5+2\sqrt{\frac{10}{7}}}$	$\frac{322-13\sqrt{70}}{900}$

Table 1: Points and weights of the univariate Gauss-Legendre quadrature rule

Exercise 5: *Numerical integration over quadrilateral element* **(7 points)**

Assume the domain $\Omega = (-1,1) \times (-1,1)$. Now the task is to compute the integral

$$\int\int_{\Omega} (x^3 y + x^2 y + 3y + 5)\mathrm{d}x\mathrm{d}y \tag{1}$$

with Gauß quadrature.

(a) Which point rule has to be used in the x direction and which in the y direction to get exact solution of (1)? (Give the minimum number of the points to be used for the two univariate rules.) Explain the answer in one sentence.

(b) Collect in a table the coordinates of the integration points and the corresponding weights of the combined rule to be used to calculate the given integral. You can use the points and the weights for the univariate rule in Table 1.

(c) Calculate the integral.

Exercise 6: *Mesh generation, adaptivity, a posterior error estimates* **(6 points)**

(a) What does it mean nonconforming triangulation? Draw an example.

(b) Explain the main idea of the strategy for choosing which triangles to refine due to Babuska and Rheinboldt by answering the following questions. Suppose there is a triangulation $\mathscr{T}_h$ with triangles $\{T_i\}_{i=1}^N$. The longest edge of the triangles is noted be h_i. Suppose that with an available error estimator we can compute the elementwise errors $\{\epsilon_i^{(1)}\}_{i=1}^N$. Let's suppose that this triangulation is coming from a uniform refinement, and we also computed the error estimators for the courser mesh, so we have for all the T_i triangles an error estimator $\{\epsilon_i^{(0)}\}_{i=1}^N$, which is the element-wise error for the triangle the subtriangles T_i were refined from.

- Main Assumptions: 1) What is our main assumption on the dependence of the elementwise error ϵ_i on the h_i diameter? 2) When do we call a mesh to be optimal?
- How do we compute the constants in the first assumption?
- There is an important a-posteriori measure, which helps choosing triangles for refinement. How do we compute this measure?
- Using the above mentioned measure, what is the criteria that chooses a triangle to be refined?

(c) What are the three components to an adaptive algorithm? (Give only a concise definition for all three.)