Technische Universität Braunschweig

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Cartoon-Texture-Noise Decomposition with Transport Norms

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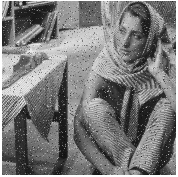
Problem

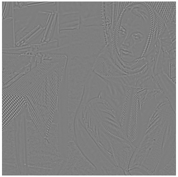
- Task: Decompose an observed image u^0 into a *cartoon part* u, a *texture part* v and a *noise part* w such that $u + v + w = u^0$.

u^0

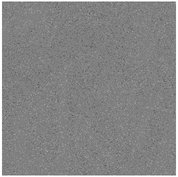


u

v



w



General Variational Approach

- Let $\Omega \subset \mathbb{R}^2$ be the *image domain* and $u^0 : \Omega \to \mathbb{R}$.
- Solve the problem

$$\min_{u,v} \quad \alpha F_u(u) + \beta F_v(v) + \gamma F_w(u^0 - u - v)$$

with positive constants α, β, γ and appropriate functionals F_u, F_v, F_w which capture discriminating features of cartoon, texture and noise.

Rudin/Osher/Fatemi Model [1992]

- The problem

$$\min_{u \in \mathrm{BV}(\Omega)} \quad \alpha \mathrm{TV}(u) + \frac{\beta}{2} \left\| u^0 - u \right\|_{L^2}^2$$

yields a decomposition into two components.

- Meyer: The ROF model does not capture texture properly.

Meyer Model [2001]

- Meyer's *G-Norm*:

$$G(\Omega) = \left\{ v \in L^2(\Omega) \mid \exists g \in L^\infty(\Omega, \mathbb{R}^2) : \operatorname{div} g = v \right\}$$
$$\|v\|_G = \inf\{\||g|\|_{L^\infty} \mid \operatorname{div} g = v\}$$

- The problem

$$\min_{(u,v)\in \mathrm{BV}(\Omega)\times G(\Omega)} \alpha \mathrm{TV}(u) + \beta \|v\|_G \quad \text{s.t.} \quad u + v = u^0$$

separates cartoon and texture properly.

- There is still no third component that allows to discriminate texture and noise.

Vese/Osher Model [2003]

- Reformulation of Meyer's model:

$$\min_{(u,g)\in \mathrm{BV}(\Omega)\times L^\infty(\Omega,\mathbb{R}^2)} \alpha \mathrm{TV}(u) + \beta \, \| |g| \|_{L^\infty} \quad \text{s.t.} \quad u + \operatorname{div} g = u^0$$

- The problem

$$\min_{(u,g)\in \mathrm{BV}(\Omega)\times L^p(\Omega,\mathbb{R}^2)} \alpha \mathrm{TV}(u) + \frac{\beta}{p} \| |g| \|_{L^p}^p + \frac{\gamma}{2} \left\| u^0 - u - \operatorname{div} \, g \right\|_{L^2}^2$$

approximates Meyer's G-Norm and relaxes the equality constraint.

- It allows for a decomposition into three components!

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Discriminating Features of Texture and Noise

- Texture features *oscillations* in the sense that local averages are close to zero, especially the total positive mass and the total negative mass are almost equal.
- *Gaussian noise* has a similar characteristic. Hence, the separation of texture and Gaussian noise is inherently difficult.
- We focus on *impulsive noise*: The total positive mass is almost equal to the total negative mass but local averages are in general not close to zero.
- Idea: One can move the positive and negative mass around to cancel each other out. This is cheap for texture and expensive for impulsive noise.

Transport Problem in Kantorovich Form [1942]

- Let μ, ν be measures on Ω with equal mass and $c : \Omega \times \Omega \to \mathbb{R}_+ \cup \{0\}$. Then,

$$\inf_{\pi} \left\{ \int_{\Omega \times \Omega} c(x, y) \, \mathrm{d}\,\pi(x, y) \mid \operatorname{proj}_1 \pi = \mu, \ \operatorname{proj}_2 \pi = \nu \right\}$$

is the *minimal cost to transport* μ *to* ν.

Wasserstein Metric [1969]

- In case $c(x,y) = d(x,y)^p$ for some metric d on $\Omega \times \Omega$ and $p \geqslant 1$,

$$W_p(\mu,\nu) = \inf_{\pi} \left\{ \int_{\Omega\times\Omega} d(x,y)^p \, \mathrm{d}\pi(x,y) \mid \mathrm{proj}_1 \pi = \mu, \mathrm{proj}_2 \pi = \nu \right\}^{\frac{1}{p}}$$

  is a metric on the space of probability measures.
- *Kantorovich-Rubinstein duality*:

$$W_1(\mu,\nu) = \sup_{f} \left\{ \int_{\Omega} f \, \mathrm{d}(\mu - \nu) \mid \mathrm{Lip}(f) \leqslant 1 \right\}$$

- $W_1(\mu,\nu)$ is infinite in case μ and ν have different total mass.

Kantorovich-Rubinstein Norm [2014]

- A variant with finite values for measures with different total mass is

$$\|\mu - \nu\|_{\mathrm{KR},\beta,\gamma} = \sup_f \left\{ \int_\Omega f \, \mathrm{d}(\mu - \nu) \mid |f| \leqslant \gamma, \ \mathrm{Lip}(f) \leqslant \beta \right\}.$$

- Dualizing again, we obtain

$$\|\mu\|_{\mathrm{KR},\beta,\gamma} = \min_{g \in W^{1,1}(\Omega;\mathrm{div})} \gamma \, \|\mu - \mathrm{div}\, g\|_{L^1} + \beta \, \||g|\|_{L^1} .$$

- $\|\mu\|_{\mathrm{KR},\beta,\gamma} = \|\mu^+ - \mu^-\|_{\mathrm{KR},\beta,\gamma}$ is the cost to transport μ^+ to μ^- w.r.t. possible mass mismatch.

G'-Norm

- A dual formulation of Meyer's G-Norm is

$$\left\|u^0 - u\right\|_G = \sup_f \left\{ \int_\Omega f(u^0 - u) \mid \||\nabla f|\|_{L^1} \leqslant 1 \right\}.$$

- Repeating the step from $W_1(\mu, \nu)$ to $\|\mu - \nu\|_{\mathrm{KR},\beta,\gamma}$ leads to

$$\left\|u^0 - u\right\|_{G',\beta,\gamma} = \sup_f \left\{ \int_\Omega f(u^0 - u) \mid \|f\|_\infty \leqslant \gamma,\ \||\nabla f|\|_{L^1} \leqslant \beta \right\}.$$

- By duality,

$$\left\|u^0 - u\right\|_{G',\beta,\gamma} = \inf_g \gamma \left\|u^0 - u - \operatorname{div} g\right\|_{L^1} + \beta \, \||g|\|_{L^\infty}.$$

Decomposition with the G'-Norm

- Meyer:

$$\min_{u,g} \quad \alpha \mathrm{TV}(u) + \beta \, |||g|||_{L^\infty} \quad \text{s.t.} \quad u + \operatorname{div} g = u^0$$

- Vese/Osher:

$$\min_{u,g} \quad \alpha \mathrm{TV}(u) + \frac{\beta}{p} \, |||g|||_{L^p}^p + \frac{\gamma}{2} \left\| u^0 - u - \operatorname{div} \, g \right\|_{L^2}^2$$

- Our model:

$$\begin{aligned} &\min_{u} \quad \alpha \mathrm{TV}(u) + \left\| u^0 - u \right\|_{G',\beta,\gamma} \\ = &\min_{u,g} \quad \alpha \mathrm{TV}(u) + \beta \, |||g|||_{L^\infty} + \gamma \left\| u^0 - u - \operatorname{div} g \right\|_{L^1} \end{aligned}$$

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Lorenz/Pock Framework

- Problem:

$$\min_x \max_y \quad G(x) + Q(x) + \langle Kx, y \rangle - F^*(y) - P^*(y)$$

- Iteration:

$$\begin{aligned}
\overline{x}^k &= x^k + \theta(x^k - x^{k-1}) \\
\overline{y}^k &= y^k + \theta(y^k - y^{k-1}) \\
x^{k+1} &= \mathrm{prox}_{\tau G}(\overline{x}^k - \tau[\nabla Q(\overline{x}^k) + K^* \overline{y}^k]) \\
y^{k+1} &= \mathrm{prox}_{\sigma F^*}(\overline{y}^k - \sigma[\nabla P^*(\overline{y}^k) - K(2x^{k+1} - \overline{x}^k)])
\end{aligned}$$

Application to Decomposition

- Dual representations of TV and $\|\cdot\|_1$ are

$$\alpha \mathrm{TV}(u) = \sup_{\phi} \quad -\langle \nabla u, \phi \rangle - I_{\||\cdot|\| \leqslant \alpha}(\phi) \quad \text{and}$$
$$\gamma \left\| u^0 - u - \operatorname{div} g \right\|_1 = \sup_{f} \quad \langle u + \operatorname{div} g - u^0, f \rangle - I_{\|\cdot\|_\infty \leqslant \gamma}(f).$$

- With

$$G(u, g) = \beta \, \||g|\|_\infty \,,$$
$$F^*(\phi, f) = I_{\||\cdot|\|_\infty \leqslant \alpha}(\phi) + I_{\|\cdot\|_\infty \leqslant \gamma}(f) + \langle u^0, f \rangle \quad \text{and}$$
$$K = \begin{bmatrix} -\nabla & 0 \\ \mathrm{Id} & \operatorname{div} \end{bmatrix}$$

the resulting iteration is...

Application to Decomposition

$$\text{extrapolate} \quad \overline{u}^k, \overline{g}^k, \overline{\phi}^k, \overline{f}^k$$

$$\begin{aligned}
u^{k+1} &= \overline{u}^k - \tau(\operatorname{div}\overline{\phi} + \overline{f}) \\
g^{k+1} &= \operatorname{prox}_{\tau\beta\,|||\cdot|||_\infty}(\overline{g}^k + \tau\nabla\overline{f}) \\
\phi^{k+1} &= \operatorname{proj}_{|||\cdot|||_\infty \leqslant \alpha}(\overline{\phi}^k - \sigma\nabla[2u^{k+1} - \overline{u}^k]) \\
f^{k+1} &= \operatorname{proj}_{\|\cdot\|_\infty \leqslant \gamma}(\overline{f}^k + \sigma[(2u^{k+1} - \overline{u}^k) + \operatorname{div}(2g^{k+1} - \overline{g}^k)] - \sigma u^0)
\end{aligned}$$

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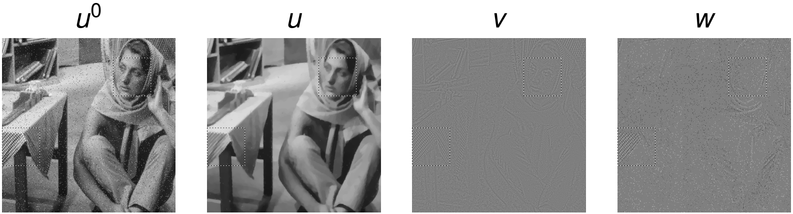
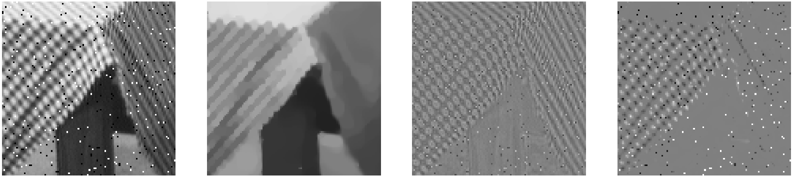
Results



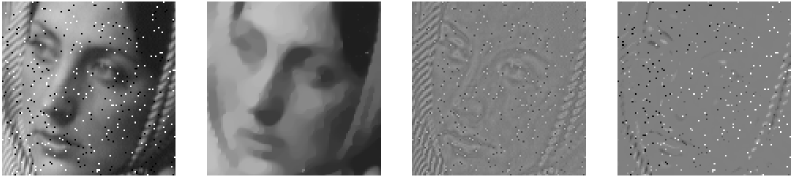
Results

u^0 u v w



Results

u^0 u v w



Reference

- Cartoon-Texture-Noise Decomposition with Transport Norms, C. Brauer and D. Lorenz, to appear in “Proceedings on Scale Space and Variational Methods”, Lecture Notes in Computer Science, 2015.

Thank you for your attention!