Introduction to PDEs and Numerical Methods
Lecture 11.
Weighted residual methods - Galerkin method, finite element method

Dr. Noemi Friedman, 10.01.2016.

Recap
Linear systems: strong form, weak form, minimization problem

If $\mathbf{A}$ is symmetric positive definite

strong form:

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

$\mathbf{x} \in R^n$

$\mathbf{A} \in R^{nxn}$

$\mathbf{b} \in R^n$

- Direct solvers
  (Gauß elimination, LU/chol decomposition)
- Iterative methods
  (Jacobi, Gauß-Seidel…)

⟺

weak form:

$$\langle \mathbf{A}\mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{b}, \mathbf{y} \rangle \; \forall \, \mathbf{y} \in R^n$$

$a(x, y)$ bilinear term

$F(y)$ linear term

- CG

residual is orthogonal w.r.t. the energy norm to the approximating Krylov subspace

equivalent ⟺ equivalent

minimization:

$$\phi(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T \mathbf{A}\mathbf{x} - \mathbf{x}^T \mathbf{b}$$

- Steepest descent
- CG

(search in the Krylov-subspace, directions are conjugates-orthogonal with respect to the engergy norm)

FROM STRONG FORM TO WEAK FORM

Simmilarly

solving PDE ⟺ **optimization of quadratic function**

Instead of solving $Lu = f$ (strong form) $Lu = f,\ u \in D_L,\ f \in H \Rightarrow u_0 \in D_L,\ Lu_0 = f$

minimize the quadratic function:

$$F(u) = \frac{1}{2}\langle Lu, u\rangle - \langle f, u\rangle$$ (weak form)

This quadratic functional attains its **stationary point** precisely where $\mathrm{Lu} = \mathrm{f}$, if L is symm (self-adjoint). $\langle Lu, v\rangle = \langle Lv, u\rangle$

↓

minimum point if L is pos. def. $\langle Lu, u\rangle \geq 0$

and only zero for u=0

FROM STRONG FORM TO WEAK FORM
Steps of formulating the weak form (recipe)

$$Lu(x) = \mathrm{f(x)}$$

1.) Multiply by test/weight function $v(x)$ and integrate

$$\langle Lu, v\rangle - \langle f, v\rangle = 0 \qquad \forall v \in V$$

$$\int Lu(x)v(x)dx - \int f(x)v(x)dx = 0 \qquad \forall v \in V$$

Check wether the PDE holds in the $v(x)$ weighted avarage sense over Ω

if it holds for all test functions then the PDE must hold

2.) Reduce order of $\langle Lu, v\rangle$ by using the integration by parts, or Green's theorem in higher dim.

3.) Apply boundary conditions

FROM STRONG FORM TO WEAK FORM
Example in 1D

Strong formulation: find $u \in C^2(I)$

$$-u''(x) + u(x) = f(x) \text{ on } I = (0,1)$$
$$u(0) = 0$$
$$u'(1) = 1$$

1) Multiply by test/weight function $v(x)$ and integrate

$$-\int_I u''(x)v(x)dx + \int_I u(x)v(x)dx = \int_I f(x)v(x)dx \qquad v(0) = 0$$

2.) Integration by parts

$$-\int_I u''(x)v(x)dx = \int_I u'(x)v'(x)dx - u'(x)v(x)|_0^1$$

3.) Apply boundary conditions ($u'(1) = 1$, $v(0) = 0$)

$$u'(x)v(x)|_0^1 = v(1)$$

FROM STRONG FORM TO WEAK FORM
Example in 1D

$$-\int_I u''(x)v(x)dx + \int_I u(x)v(x)dx = \int_I f(x)v(x)dx$$

$$= \int_I u'(x)v'(x)dx - v(1)$$

Weak formulation: For $V = \{H^1(I) \,|\, u(0) = 0\}$ find $u \in V$ such that

$$\int_I u'(x)v'(x)dx + \int_I u(x)v(x)dx = \int_I f(x)v(x)dx + v(1) \quad \forall v \in V$$

Abstract setting:

$$V = \{H^1(I) \,|\, u(0) = 0\}, \quad a(u,v) = \int_I u'(x)v'(x)dx + \int_I u(x)v(x)dx, \quad l(v) = \int_I f(x)v(x)dx$$

$$a(u,v) = l(v) \quad \forall v \in V.$$

Existence and uniqueness of the solution of BVPs

Strong form:

$$Lu(\mathbf{x}) = f(\mathbf{x})$$

Weak form:

$$\langle Lu, v\rangle = \langle f, v\rangle$$

$\forall v \in ?$

$u \in ?$

$a(u, v)$ bilinear term

$l(v)$ linear term

Does the solution exists?
Does it have a unique solution?

In accordance to the Lax-Milgram Lemma if:

$l(\cdot)$	bounded, linear functional
$a(\cdot,\cdot)$	bounded, V-elliptic bilinear functional and
V	a Hilbert space

- solution u exists
- unique solution $u \in V$
- *+solution u depends continiously on f*

For a specific BVP one has to find the right Hilbert space (Sobolev space or L2 space) where the conditions for $a(\cdot,\cdot)$ and $l(\cdot)$ are satisfied, and then we know, that in that space we have a unique solution!

Dense and complete spaces

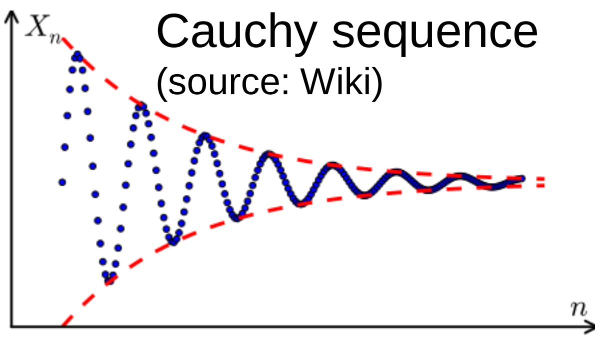
A subset W of a space V is called **dense (in V)** if every point v in V either belongs to W or arbitrarily "close" to a member of W

Cauchy sequence

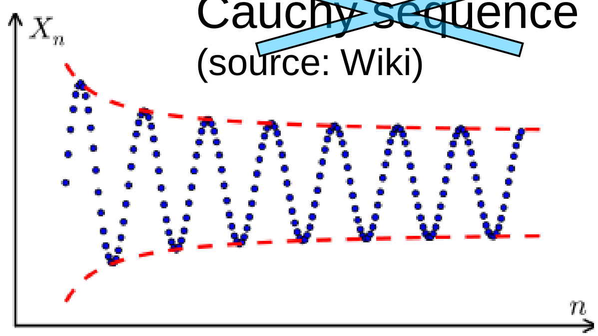
$x_1, x_2, x_3, \ldots$

For every positive real number ε, there is a positive integer N such that for all m, n > N

$$\|x_m - x_n\| < \varepsilon$$



Cauchy sequence
(source: Wiki)



~~Cauchy sequence~~
(source: Wiki)

A **normed space** V is called **complete** (or a **Cauchy space**) if every Cauchy sequence of points in V has a limit that is also in V or, alternatively, if every Cauchy sequence in V converges in V

Some more fundamentals of functional analysis
Important vector spaces

Banach space: complete, normed vector space examples:

- Lp spaces
- Hilbert space with norm $\|x\|_H = \sqrt{\langle x, x \rangle},$

Hilbert space: complete, inner product space examples

- L_2 space
- Sobolev spaces

Lebesque space (Lp)

with the norm: $\|f\|_p = \left(\int_\Omega f(x)^p \mathrm{dx}\right)^{\frac{1}{p}}$

L2 space (square-integrable functions): $\int_\Omega f(x)^2 \mathrm{dx} < \infty$

inner product: $\langle f, g \rangle = \int_\Omega f(x) g(x) \mathrm{dx}$ norm: $\|f\|_2 = \sqrt{\langle f, f \rangle} = \sqrt{\int_\Omega f(x)^2 \mathrm{dx}}$

Sobolev space (Hp)
norm: combination of Lp-norms of the function itself and its derivatives up to p order
derivatives: weak derivatives→ complete space →Banach space
Examples:

$$H_1(\mathbb{R}) = \{u | u, u' \in L_2(\mathbb{R})\}$$

$$H_1(\mathbb{R}^2) = \left\{u \middle| u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \in L_2(\mathbb{R})\right\}$$

$$H_2(\mathbb{R}) = \{u | u, u', u'' \in L_2(\mathbb{R})\}$$

$$H_2(\mathbb{R}^2) = \left\{u \middle| u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}, \frac{\partial^2 u}{\partial x \partial y} \in L_2(\mathbb{R})\right\}$$

$${H_p}^0(\mathbb{R}) = \left\{u \middle| u, u' \in L_2(\mathbb{R}), u|_{\Gamma_D} = 0\right\}$$

Some more fundamentals of functional analysis
Bilinear functionals, and its properties

A functional a, mapping from UxV into a scalar $\mathbb{R}$, is **bilinear** if

$$a(u+w,v) = a(u,v) + a(w,v) \quad \forall u,w \in U,\ v \in V$$

$$a(u,v+w) = a(u,v) + a(u,w), \quad \forall u \in U,\ v,w \in V$$

$$a(\alpha u,v) = \alpha a(u,v) \quad \forall \alpha \in \mathbb{R}, u \in U, v \in V$$

$$a(u,\alpha v) = \alpha a(u,v) \quad \forall \alpha \in \mathbb{R}, u \in U, v \in V.$$

A bilinear functional a, is **symmetric** if

$$a(u,v) = a(v,u)$$

A bilinear functional a, is **positive definite** if

$$a(u,u) \geq 0,\ a(u,u) = 0 \text{ only if } u = 0$$

A bilinear functional a, is **bounded** if there exists an $M > 0$ such that $\forall\ u \in U$ *and* $v \in V$

$$a(u,v) \leq M\|u\|\|u\|$$

A bilinear functional a, is **V-elliptic** if there exists a $\delta > 0$ such that $\forall\ u \in U$

$$a(u,u) \geq \delta\|u\|^2$$

Some more fundamentals of functional analysis
Lax-Milgram Lemma

Let

$a(\cdot,\cdot)$ be a bounded, V-elliptic bilinear functional and

V a Hilbert space

Then for any $f \in V^*$ (that is, for any linear, bounded functionals mapping from V to $\mathbb{R}$) there is a unique solution $u \in V$ to the equation:

$$a(u,v) = f(v)$$

and moreover, this unique solution u depends continiously on f:

$$\|u\|_V \leq \frac{1}{\delta}\|\mathrm{f}(u)\|_{V*}$$

Energy inner product, energy norm

- If $a(\cdot,\cdot)$ is a bilinear functional that is
- bounded
- V-elliptic
- symmetric

bounded, V-elliptic → positive definite

then it is an inner product, called the **energy inner** product:

$$\langle u, v \rangle_E = a(u, v)$$

The corresponding induced norm is called the energy norm:

$$\|u\|_E = \sqrt{a(u, u)}$$

Existence and uniqueness of the solution of BVPs – examples

1. Poisson equation:

$$-\Delta u(\mathbf{x}) = f(\mathbf{x}) \Rightarrow \quad \mathrm{F}(v) = \int f(\mathbf{x})v(\mathbf{x})\boldsymbol{d}\mathbf{x}, \quad \mathrm{a}(\cdot,\cdot) = \int \nabla u(\mathbf{x})\nabla v(\mathbf{x})\boldsymbol{d}\mathbf{x}$$

$\mathrm{F}(\cdot)$	linear functional, in H_1^0:	bounded
$\mathrm{a}(\cdot,\cdot)$	bilinear functional, in H_1^0:	bounded, V-elliptic

⇒
- unique solution $u \in H_1^0$
- $v \in H_1^0$

$\mathrm{a}(\cdot,\cdot)$	bilinear functional, in L_2:	not bounded
$\mathrm{a}(\cdot,\cdot)$	bilinear functional, in H_1:	not V-elliptic

⇒ we have to narrow down the space in which we look for the solution, because we can not prove that there is a unique solution in L2 or in H1

2. Plate equation

$$-\Delta\Delta u(\mathbf{x}) = f(\mathbf{x}) \Rightarrow \quad \mathrm{F}(v) = \int f(\mathbf{x})v(\mathbf{x})\boldsymbol{d}\mathbf{x}, \quad \mathrm{a}(\cdot,\cdot) = \int \Delta u(\mathbf{x})\Delta v(\mathbf{x})\boldsymbol{d}\mathbf{x}$$

$\mathrm{F}(\cdot)$	linear functional, in H_2^E:	bounded
$\mathrm{a}(\cdot,\cdot)$	bilinear functional, in H_2^E:	bounded, V-elliptic

⇒
- unique solution $u \in H_2^E$
- $v \in H_2^E$

Discretisation

Further simplifications (discretize to finite dimensional space)

- Approximate the solution with some basis/shape functions:

$$u(x) = \sum_i u_i \Phi_i(x)$$

- Instead of solving it for all $v(x) \in V$, select finite subspace for the weighting functions:

$$v(x) = \sum_i u_i \varphi_i(x)$$

How to choose the subspace? How to choose the weighting functions $v(x)$?

- True solution can be well approximated by an element of the subspace
- Efficient computation

Bubnov-Galerkin method $(\Phi_i = \varphi_i)$

FEM: Galerkin method with subspace of *piecewise polynomial functions*

Petrov-Galerkin method $(\Phi_i \neq \varphi_i)$

Pointwise collocation $\varphi_i = \delta(x - x_i)$

Subdomain collocation $\varphi_i = \chi_{\Omega_i}$