Technische
Universität
Braunschweig

Introduction to PDEs and Numerical Methods

Lecture 6:

Numerical solution of the heat equation with FD method: method of lines, Euler forward, Euler backward, the Theta method, and their stability

Dr. Noemi Friedman, 22.11.2017.

Overview of the course

- Introduction (definition of PDEs, classification, basic math, introductory examples of PDEs)

- Analytical solution of elementary PDEs (Fourier series/transform, separation of variables, Green's function)

- Numerical solutions of PDEs:

    - Finite difference method

    - Finite element method

Overview of this lecture

- Finite difference operators
- The heat equation
    - Analytical solution
    - Semidiscretization- Method of Lines (spatial discretization)
    - Time discretization
        - Euler foward
        - Euler backward
        - The theta-method

    - Consisteny, stability, convergence
    - Heat equation in higher dimension

Numerical solution of the heat equation
We only analyse now the hom. equation - why can we do that?

$$\boxed{\frac{d}{dt}\boldsymbol{u} = \boldsymbol{A}\boldsymbol{u}(t) + \boldsymbol{f}}$$

Instead of analyzing stability of the inhomogenous case, we discretize the homogenous one. We can do that, because of the following.
Let's take a stationary function $\boldsymbol{u_0}$ for which the equation:

$$\frac{d}{dt}\boldsymbol{u_0} = \boldsymbol{A}\boldsymbol{u_0} + \boldsymbol{f} = 0$$

holds.

And let's suppose, that the solution can be written in the form $\boldsymbol{u}(t) = \boldsymbol{u_0} + \boldsymbol{v}(t)$

r.h.s:

$$\boldsymbol{A}\boldsymbol{u}(t) + \boldsymbol{f} = \underbrace{\boldsymbol{A}\boldsymbol{u_0} + \boldsymbol{f}}_{=0} + \boldsymbol{A}\boldsymbol{v}(t) = \boldsymbol{A}\boldsymbol{v}(t)$$

l.h.s:

$$\frac{d}{dt}\boldsymbol{u}(t) = \frac{d}{dt}\big(\boldsymbol{u_0} + \boldsymbol{v}(t)\big) = \frac{d}{dt}\boldsymbol{v}(t)$$

$$\boxed{\frac{d}{dt}\boldsymbol{v}(t) = \boldsymbol{A}\boldsymbol{v}(t)}$$

Heat equation
Comparism of the solutions: analytical and method of lines

Homogeneous equation (no internal source term): $$\boxed{\frac{d}{dt}\boldsymbol{u} = \boldsymbol{A}\boldsymbol{u}(t)}$$

Analytical solution with hom. BC:

$$\boxed{u(x,t) = \sum_j d_j \, e^{-\omega_j^2 \beta t} \sin(\omega_j x) \quad \omega_j = \frac{j\pi}{l} \quad j = 0,1,2..}$$

⇨ as $t \to \infty \quad u \to 0$

With method of lines:

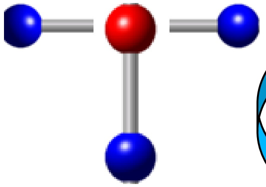
$$\boxed{\boldsymbol{u}(t) = \sum_i c_i e^{\lambda_i (t-t_0)} \boldsymbol{v_i}}$$

$$\lambda_i = \underbrace{\frac{2\beta^2}{h^2}}_{>0} \left(\underbrace{\cos\left(\frac{i\pi}{N}\right)}_{<1} - 1 \right)$$

⇨ as $t \to \infty \quad u \to 0$

Numerical solution of the heat equation
2.) Euler forward - explicit

Euler forward: approximate time derivate with the forward difference



$$\boxed{\frac{\partial u_j(t)}{\partial t}} = \frac{\beta^2}{h^2}\left(u_{j-1} - 2u_j + u_{j+1}\right) + O(h^2)$$

$$\boxed{\frac{d}{dt}\mathbf{u}(t)} = \mathbf{A}\mathbf{u}(t)$$

$$\boxed{\dot{u}_{j,n} = \frac{u_{j,n+1} - u_{j,n}}{\Delta t} + O(\Delta t)}$$

$$\boxed{\dot{\boldsymbol{u}}_n = \frac{\boldsymbol{u}_{n+1} - \boldsymbol{u}_n}{\Delta t} = \mathbf{A}\mathbf{u}_\mathrm{n}}$$

$$\frac{u_{j,n+1} - u_{j,n}}{\Delta t} = \frac{\beta^2}{h^2}\left(u_{j-1,n} - 2u_{j,n} + u_{j+1,n}\right) + O(h^2) + O(\Delta t)$$

$$\mathbf{u}_{n+1} = \mathbf{u}_n + \Delta t\,\mathbf{A}\mathbf{u}_n$$

$$= \underbrace{(\mathbf{I} + \Delta t\,\mathbf{A})}_{\mathbf{B}}\mathbf{u}_n$$

Stable? (does it give decaying solution?)

$$B = \frac{-\Delta t \cdot \beta^2}{h^2}\begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & & & \\ & & & 2 & -1 & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix}$$

Numerical solution of the heat equation
2.) Euler forward – d-tour on eigenvalues and eigenvectors

Stable? (does it give decaying solution?)

$\mathbf{u_{n+1}} = \mathbf{B}\mathbf{u_n}$ $\quad\mathbf{u_{n+2}} = \mathbf{B}^2\mathbf{u_n}$ $\quad\mathbf{u_{n+3}} = \mathbf{B}^3\mathbf{u_n}$ $\quad\mathbf{u_{n+k}} = \mathbf{B}^k\mathbf{u_n}$

Some discussion about egienvalues and eigenvectors:

$$\mathbf{A}\mathbf{v_i} = \lambda_i \mathbf{v_i}$$

$$\mathbf{AV} = \mathbf{DV}$$

$$\mathbf{V}^{-1}\mathbf{AV} = \mathbf{D}$$

⇒ $\mathbf{V}^{-1}\mathbf{AV}$ is a diagonal matrix, with the eigenvalues in the diagonal.

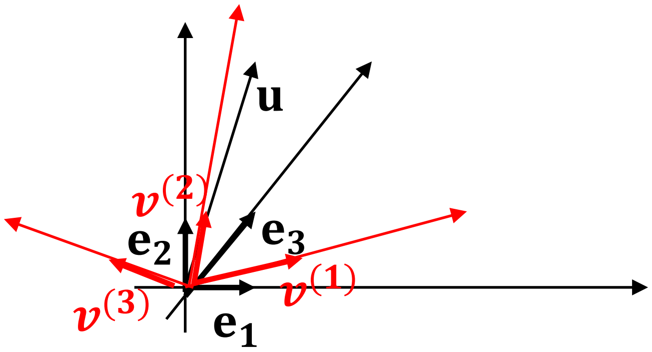
$$\mathbf{D} = \begin{bmatrix} \lambda_1 & & & & & \\ & \lambda_2 & & & & \\ & & \ddots & & & \\ & & & \ddots & & \\ & & & & \lambda_{n-1} & \\ & & & & & \lambda_n \end{bmatrix}$$

$$\begin{array}{cc} & \begin{array}{cccccc} \mathbf{v_1} & \mathbf{v_2} & & & \mathbf{v_{n-1}} & \mathbf{v_n} \end{array} \\ \mathbf{V} = & \left[\begin{array}{cccccc} | & | & \dots & & | & | \\ | & | & & \dots & | & | \\ | & | & & & | & | \end{array}\right] \end{array}$$

Eigenvalues, eigenvectors
change of basis (coordinate system)

How to write an arbitrary vector, $\mathbf{u}$, in a new basis $\boldsymbol{v}^{(1)}$, $\boldsymbol{v}^{(2)}$, $\boldsymbol{v}^{(3)}$

What is $\mathbf{V}^{-1}\mathbf{A}\mathbf{V}$?



$$\mathbf{u} = u_1\mathbf{e_1} + u_2\mathbf{e_2} + u_3\mathbf{e_3}$$

$$\mathbf{u} = \tilde{u}_1\boldsymbol{v}^{(1)} + \tilde{u}_2\boldsymbol{v}^{(2)} + \tilde{u}_3\boldsymbol{v}^{(3)}$$

$$\boldsymbol{v}^{(1)} = v_1^{(1)}\mathbf{e_1} + v_2^{(1)}\mathbf{e_2} + v_3^{(1)}\mathbf{e_3}$$

$$\boldsymbol{v}^{(2)} = v_1^{(2)}\mathbf{e_1} + v_2^{(2)}\mathbf{e_2} + v_3^{(2)}\mathbf{e_3}$$

$$\boldsymbol{v}^{(3)} = v_1^{(3)}\mathbf{e_1} + v_2^{(3)}\mathbf{e_2} + v_3^{(3)}\mathbf{e_3}$$

$$\begin{aligned}\mathbf{u} = \tilde{u}_1&\left(v_1^{(1)}\mathbf{e_1} + v_2^{(1)}\mathbf{e_2} + v_3^{(1)}\mathbf{e_3}\right) + \cdots\\ +\tilde{u}_2&\left(v_1^{(2)}\mathbf{e_1} + v_2^{(2)}\mathbf{e_2} + v_3^{(2)}\mathbf{e_3}\right) + \cdots\\ +\tilde{u}_3&\left(v_1^{(3)}\mathbf{e_1} + v_2^{(3)}\mathbf{e_2} + v_3^{(3)}\mathbf{e_3}\right) + \cdots\end{aligned}$$

$$\Rightarrow$$

$$\begin{aligned}\mathbf{u} = \mathbf{e_1}&\overbrace{\left(\tilde{u}_1 v_1^{(1)} + \tilde{u}_2 v_1^{(2)} + \tilde{u}_3 v_1^{(3)}\right)}^{u_1} + \cdots\\ \mathbf{e_2}&\overbrace{\left(\tilde{u}_1 v_2^{(1)} + \tilde{u}_2 v_2^{(2)} + \tilde{u}_3 v_2^{(3)}\right)}^{u_2} + \cdots\\ \mathbf{e_3}&\underbrace{\left(\tilde{u}_1 v_3^{(1)} + \tilde{u}_2 v_3^{(2)} + \tilde{u}_3 v_3^{(3)}\right)}_{u_3} + \cdots\end{aligned}$$

Eigenvalues, eigenvectors
change of basis (coordinate system)

Some discussion about egienvalues and eigenvectors:

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{array}{c} \begin{array}{ccc} \boldsymbol{v}^{(1)} & \boldsymbol{v}^{(2)} & \boldsymbol{v}^{(3)} \\ \downarrow & \downarrow & \downarrow \end{array} \\ \begin{bmatrix} v_1^{(1)} & v_1^{(2)} & v_1^{(3)} \\ v_2^{(1)} & v_2^{(2)} & v_2^{(3)} \\ v_3^{(1)} & v_3^{(2)} & v_3^{(3)} \end{bmatrix} \end{array} \begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \\ \tilde{u}_3 \end{bmatrix} \Longleftarrow$$

$$\begin{aligned} \mathbf{u} = \mathbf{e_1} & \overbrace{\left(\tilde{u}_1 v_1^{(1)} + \tilde{u}_2 v_1^{(2)} + \tilde{u}_3 v_1^{(3)}\right)}^{u_1} + \cdots \\ \mathbf{e_2} & \overbrace{\left(\tilde{u}_1 v_2^{(1)} + \tilde{u}_2 v_2^{(2)} + \tilde{u}_3 v_2^{(3)}\right)}^{u_2} + \cdots \\ \mathbf{e_3} & \underbrace{\left(\tilde{u}_1 v_3^{(1)} + \tilde{u}_2 v_3^{(2)} + \tilde{u}_3 v_3^{(3)}\right)}_{u_3} + \cdots \end{aligned}$$

$$\Downarrow$$

$$\mathbf{u} = \mathbf{V}\tilde{\mathbf{u}} \Longrightarrow \tilde{\mathbf{u}} = \mathbf{V}^{-1}\mathbf{u}$$

Let’s consider now the respresentation of an operator in the new basis:

$$\mathbf{A}\mathbf{u} = \mathbf{b}$$

$$\mathbf{A}\mathbf{V}\tilde{\mathbf{u}} = \mathbf{V}\tilde{\mathbf{b}}$$

$$\mathbf{V}^{-1}\mathbf{A}\mathbf{V}\tilde{\mathbf{u}} = \mathbf{V}^{-1}\mathbf{V}\tilde{\mathbf{b}}$$

$\mathbf{V}^{-1}\mathbf{A}\mathbf{V}$ is the representationa of the $\mathbf{A}$ operator in the new basis defined by $\boldsymbol{v}^{(1)}$, $\boldsymbol{v}^{(2)}$, $\boldsymbol{v}^{(3)}$

$$\overbrace{\mathbf{V}^{-1}\mathbf{A}\mathbf{V}}\tilde{\mathbf{u}} = \tilde{\mathbf{b}}$$

Numerical solution of the heat equation
2.) Euler forward – d-tour on eigenvalues and eigenvectors

$\mathbf{V}^{-1}\mathbf{A}\mathbf{V}$ is the representationa of the $\mathbf{A}$ operator in the new basis defined by $\boldsymbol{v}^{(1)}, \boldsymbol{v}^{(2)}, \boldsymbol{v}^{(3)}$

If $\boldsymbol{v}^{(1)}, \boldsymbol{v}^{(2)}, \boldsymbol{v}^{(3)}$ are the eigenvectors of $\mathbf{A}$, this representation is a diagonal matrix.

Let's go back to the problem of the Euler forward method used for solving the heat equation

---

Stable? (does it give decaying solution?)

$$\mathbf{u}_{\mathrm{n+1}} = \mathbf{B}\mathbf{u}_{\mathrm{n}} \qquad \mathbf{u}_{\mathrm{n+2}} = \mathbf{B}^2\mathbf{u}_{\mathrm{n}} \qquad \ldots \qquad \mathbf{u}_{\mathrm{n+k}} = \mathbf{B}^k\mathbf{u}_{\mathrm{n}}$$

Let's check instead in the basis defined by the eigenvectors: $\tilde{\mathbf{u}}_{\mathrm{i}} = \mathbf{V}^{-1}\boldsymbol{u}_{\mathrm{i}}$

$$\tilde{\mathbf{u}}_{\mathrm{n+1}} = \mathbf{V}^{-1}\mathbf{B}\mathbf{V}\tilde{\mathbf{u}}_{\mathrm{n}} \qquad \tilde{\mathbf{u}}_{\mathrm{n+2}} = \mathbf{V}^{-1}\mathbf{B}^2\boldsymbol{V}\tilde{\mathbf{u}}_{\mathrm{n}} \qquad \tilde{\mathbf{u}}_{\mathrm{n+k}} = \mathbf{V}^{-1}\mathbf{B}^k\boldsymbol{V}\tilde{\mathbf{u}}_{\mathrm{n}}$$

$$\mathbf{V}^{-1}\mathbf{B}^k\boldsymbol{V} = \mathbf{V}^{-1}\boldsymbol{B}\boldsymbol{B}\ldots\boldsymbol{B}\boldsymbol{B}\boldsymbol{V} = \mathbf{V}^{-1}\boldsymbol{B}\boldsymbol{V}\mathbf{V}^{-1}\boldsymbol{B}\boldsymbol{V}\mathbf{V}^{-1}\ldots\boldsymbol{V}\underbrace{\mathbf{V}^{-1}\boldsymbol{B}\boldsymbol{V}}_{\boldsymbol{D}}\underbrace{\mathbf{V}^{-1}\boldsymbol{B}\boldsymbol{V}}_{\boldsymbol{D}} = \boldsymbol{D}^k = \begin{bmatrix} \lambda_1^k & & & \\ & \lambda_2^k & & \\ & & \ddots & \\ & & & \lambda_n^k \end{bmatrix}$$

Numerical solution of the heat equation
2.) Euler forward – stability analysis

Stable? (does it give decaying solution?)

$$B = \frac{-\Delta t \cdot \beta^2}{h^2} \begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & & & \\ & & & 2 & -1 & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix}$$

When for all the eigenvalues $|\lambda_i| < 1$ then response is decaying with time.

$$a = 1 - 2\frac{\Delta t \cdot \beta^2}{h^2} \qquad b = \frac{\Delta t \cdot \beta^2}{h^2}$$

$$\lambda_i = a + 2b\cos\left(\frac{i\pi}{N}\right) = 1 - 2\frac{\Delta t \beta^2}{h^2}\left(1 - \cos\frac{i\pi}{N}\right)$$

$$\left|1 - 2\frac{\Delta t \beta^2}{h^2}\left(1 - \cos\frac{i\pi}{N}\right)\right| < 1$$

$$i = 1..N-1$$

Numerical solution of the heat equation
2.) Euler forward – stability analysis

Stable? (does it give decaying solution?)

When for all the eigenvalues $|\lambda_i| < 1$ then response is decaying with time.

$$\left|1-2\frac{\Delta t\beta^2}{h^2}\left(1-\cos\frac{i\pi}{N}\right)\right|<1$$

$$i=1..N-1$$

$$2\frac{\Delta t\beta^2}{h^2}\left(1-\cos\frac{i\pi}{N}\right)<1$$

allways satisfied

$$1-2\underbrace{\frac{\Delta t\beta^2}{h^2}\left(1-\cos\frac{i\pi}{N}\right)}_{<1}<1$$

$$2\frac{\Delta t\beta^2}{h^2}\left(1-\cos\frac{i\pi}{N}\right)\succ 1$$

0<   ≻ 2
Max value: 2

$$1-2\underbrace{\frac{\Delta t\beta^2}{h^2}\left(1-\cos\frac{i\pi}{N}\right)}_{>1}>-1$$

$$2\frac{\Delta t\beta^2}{h^2}\left(1-\cos\frac{i\pi}{N}\right)<2$$

$$\overbrace{\left(1-\cos\frac{i\pi}{N}\right)}<\frac{h^2}{\Delta t\beta^2}$$

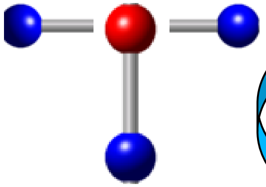
$$2<\frac{h^2}{\Delta t\beta^2}$$

$$\boxed{\Delta t<\frac{h^2}{2\beta^2}}$$

Method is not unconditionally stable, only stable if criteria is satisfied!

Numerical solution of the heat equation
2.) Euler forward – summary

Euler forward: approximate time derivate with the forward difference



$$\frac{\partial u_j(t)}{\partial t} = \frac{\beta^2}{h^2}\left(u_{j-1} - 2u_j + u_{j+1}\right) + O(h^2)$$

$$\frac{d}{dt}\mathbf{u}(t) = \mathbf{A}\mathbf{u}(t)$$

$$\boxed{\dot{u}_{j,n} = \frac{u_{j,n+1} - u_{j,n}}{\Delta t} + O(\Delta t)}$$

$$\boxed{\dot{\boldsymbol{u}}_n = \frac{\boldsymbol{u}_{n+1} - \boldsymbol{u}_n}{\Delta t} = \mathbf{A}\mathbf{u}_n}$$

$$\frac{u_{j,n+1} - u_{j,n}}{\Delta t} = \frac{\beta^2}{h^2}\left(u_{j-1,n} - 2u_{j,n} + u_{j+1,n}\right) + O(h^2) + O(\Delta t)$$

$$\mathbf{u}_{n+1} = \mathbf{u}_n + \Delta t\,\mathbf{A}\mathbf{u}_n$$

$$= \underbrace{(\mathbf{I} + \Delta t\,\mathbf{A})}_{\mathbf{B}}\mathbf{u}_n$$

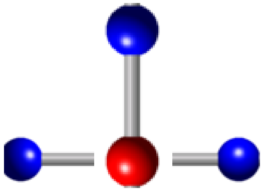
Stable? (does it give decaying solution?)

the absolut values of the eigenvalues of matrix **B** can not be greater than one.

$$\lambda_j = 1 - 2\frac{\beta^2 \Delta t}{h^2}\left(1 - \cos\frac{(N-1)\pi}{N}\right) \qquad |\lambda_j| \le 1 \;\Longrightarrow\; \Delta t < \frac{h^2}{2\beta^2}$$

Numerical solution of the heat equation
3.) Euler backward (implicit) – derivation

Euler backward: approximate time derivate with the backward difference



$$\frac{\partial u_j(t)}{\partial t} = \frac{\beta^2}{h^2}\left(u_{j-1} - 2u_j + u_{j+1}\right) + O(h^2)$$

$$\frac{d}{dt}\mathbf{u}(t) = \mathbf{A}\mathbf{u}(t)$$

$$\dot{u}_{j,n+1} = \frac{u_{j,n+1} - u_{j,n}}{\Delta t} + O(\Delta t)$$

$$\dot{\mathbf{u}}_{n+1} = \frac{\mathbf{u}_{n+1} - \mathbf{u}_n}{\Delta t} = \mathbf{A}\mathbf{u}_{n+1}$$

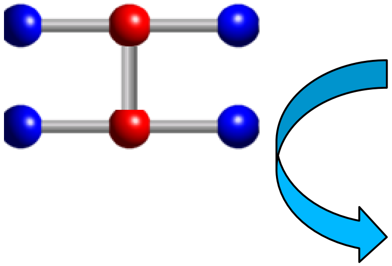
$$\frac{u_{j,n+1} - u_{j,n}}{\Delta t} = \frac{\beta^2}{h^2}\left(u_{j-1,n+1} - 2u_{j,n+1} + u_{j+1,n+1}\right) + O(h^2) + O(\Delta t)$$

$$\mathbf{u}_n = \mathbf{u}_{n+1} - \Delta t\mathbf{A}\mathbf{u}_{n+1} = (\mathrm{I} - \Delta t\mathbf{A})\mathbf{u}_{n+1}$$

(Stability criteria: unconditionaly stable, to be shown later)

Numerical solution of the heat equation
4.) Theta method (implicit) – derivation

Theta method (Crank-Nicolson) $\boxed{\frac{\partial u_j(t)}{\partial t}} = \frac{\beta^2}{h^2}\left(u_{j-1} - 2u_j + u_{j+1}\right) + O(h^2)$ $\frac{d}{dt}\mathbf{u}(t) = \mathbf{A}\mathbf{u}(t)$



$$\boxed{\dot{u}_{j,n+\theta} = \frac{u_{j,n+1} - u_{j,n}}{\Delta t} + O(\Delta t^p)}$$

$$\dot{u}_{j,n} = \frac{u_{j,n+1} - u_{j,n}}{\Delta t} + O(\Delta t)$$

Euler f.

$$\dot{\boldsymbol{u}}_n = \frac{\boldsymbol{u}_{n+1} - \boldsymbol{u}_n}{\Delta t} = \mathbf{A}\mathbf{u}_n$$

$$\dot{u}_{j,n+1} = \frac{u_{j,n+1} - u_{j,n}}{\Delta t} + O(\Delta t)$$

Euler b.

$$\dot{\mathbf{u}}_{n+1} = \frac{\mathbf{u}_{n+1} - \mathbf{u}_n}{\Delta t} = \mathbf{A}\mathbf{u}_{n+1}$$

$$\frac{u_{j,n+1} - u_{j,n}}{\Delta t} = \theta \frac{\beta^2}{h^2}\left(u_{j-1,n+1} - 2u_{j,n+1} + u_{j+1,n+1}\right) +$$
$$+(1-\theta)\frac{\beta^2}{h^2}\left(u_{j-1,n} - 2u_{j,n} + u_{j+1,n}\right) + O(h^2) + O(\Delta t^p)$$

$$\boxed{\dot{\mathbf{u}}_{\mathrm{n}+\theta} = \frac{\mathbf{u}_{\mathrm{n}+1} - \mathbf{u}_{\mathrm{n}}}{\Delta t} = \theta\mathbf{A}\mathbf{u}_{\mathrm{n}+1} + (1-\theta)\mathbf{A}\mathbf{u}_{\mathrm{n}}}$$

$$\underbrace{(\mathbf{I} - \theta\Delta t\mathbf{A})}_{\mathbf{B}_1}\mathbf{u}_{\mathrm{n}+1} = \underbrace{(\mathbf{I} + (1-\theta)\Delta t\mathbf{A})}_{\mathbf{B}_2}\mathbf{u}_{\mathrm{n}}$$

$\theta = 0$ Euler forward

$\theta = 1$ Euler backward

Numerical solution of the heat equation
4.) Theta method (implicit) – stability analysis

Stability criteria:

$$\mathbf{B_1} = (\mathbf{I} - \theta\Delta t\mathbf{A})$$

$$\mathbf{B_2} = (\mathbf{I} + (1-\theta)\Delta t\mathbf{A})$$

$$\mathbf{B_1}\mathbf{u}_{\mathrm{n+1}} = \mathbf{B_2}\mathbf{u}_{\mathrm{n}}$$

1) $\mathbf{B_1}$ and $\mathbf{B_2}$ are both tridiagonal sym. matrices

➡ $\mathbf{B_1}$ and $\mathbf{B_2}$ have the same eigenvectors

$$\boldsymbol{v_i} = \begin{bmatrix} v_i^1 \\ \vdots \\ v_i^{N-1} \end{bmatrix} \qquad v_i^k = \sin\left(\frac{ik\pi}{N}\right)$$

2) The representation in the basis defined by the eigenvectors

$$\underbrace{\mathbf{V}^{-1}\mathbf{B_1}\boldsymbol{V}}_{\mathbf{D_1}}\,\tilde{\mathbf{u}}_{\mathrm{n+1}} = \underbrace{\mathbf{V}^{-1}\mathbf{B_2}\boldsymbol{V}}_{\mathbf{D_2}}\tilde{\mathbf{u}}_{\mathrm{n}}$$

$$\begin{bmatrix} \lambda_1(\mathbf{B_1}) & & & \\ & \lambda_2(\mathbf{B_1}) & & \\ & & \ddots & \\ & & & \lambda_n(\mathbf{B_1}) \end{bmatrix}\tilde{\mathbf{u}}_{\mathrm{n+1}} = \begin{bmatrix} \lambda_1(\mathbf{B_2}) & & & \\ & \lambda_2(\mathbf{B_2}) & & \\ & & \ddots & \\ & & & \lambda_n(\mathbf{B_2}) \end{bmatrix}\tilde{\mathbf{u}}_{\mathrm{n}}$$

Numerical solution of the heat equation
4.) Theta method (implicit) – stability analysis

Stability criteria:

$$\mathbf{B_1 u_{n+1}} = \mathbf{B_2 u_n}$$

$$\mathbf{B_1} = (\mathbf{I} - \theta \Delta t \mathbf{A})$$

$$\mathbf{B_1} = \begin{bmatrix} a_1 & b_1 & & \\ b_1 & a_1 & b_1 & \\ & b_1 & a_1 & \\ & & & \ddots \end{bmatrix}$$

$$\tilde{\mathbf{u}}_{n+1} = \begin{bmatrix} \frac{\lambda_1(\mathbf{B_2})}{\lambda_1(\mathbf{B_1})} & & & \\ & \frac{\lambda_2(\mathbf{B_2})}{\lambda_2(\mathbf{B_1})} & & \\ & & \ddots & \\ & & & \frac{\lambda_n(\mathbf{B_2})}{\lambda_n(\mathbf{B_1})} \end{bmatrix} \tilde{\mathbf{u}}_n$$

$$\mathbf{B_2} = (\mathbf{I} + (1-\theta) \Delta t \mathbf{A})$$

$$\mathbf{B_2} = \begin{bmatrix} a_2 & b_2 & & \\ b_2 & a_2 & b_2 & \\ & b_2 & a_2 & \\ & & & \ddots \end{bmatrix}$$

$$\lambda_i(\mathbf{B_1}) = a_1 + 2b_1 \cos\left(\frac{i\pi}{N}\right) = 1 + \frac{2\beta^2 \Delta t}{h^2} \theta \left(1 - \cos\left(\frac{i\pi}{N}\right)\right)$$

$$\lambda_i(\mathbf{B_2}) = a_2 + 2b_2 \cos\left(\frac{i\pi}{N}\right) = 1 - \frac{2\beta^2 \Delta t}{h^2} (1-\theta) \left(1 - \cos\left(\frac{i\pi}{N}\right)\right)$$

Numerical solution of the heat equation
4.) Theta method (implicit) – stability analysis

$$\tilde{\mathbf{u}}_{n+1} = \begin{bmatrix} \frac{\lambda_1(\mathbf{B}_2)}{\lambda_1(\mathbf{B}_1)} & & & \\ & \frac{\lambda_2(\mathbf{B}_2)}{\lambda_2(\mathbf{B}_1)} & & \\ & & \ddots & \\ & & & \frac{\lambda_n(\mathbf{B}_2)}{\lambda_n(\mathbf{B}_1)} \end{bmatrix} \tilde{\mathbf{u}}_n$$

$$i = 1..\overbrace{N-1}^{n}$$

$$\left|\frac{\lambda_i(\mathbf{B}_2)}{\lambda_i(\mathbf{B}_1)}\right| = \left| \frac{1 - \overbrace{\frac{2\beta^2\Delta t}{h^2}}^{2r}(1-\theta)\overbrace{\left(1-\cos\left(\frac{i\pi}{N}\right)\right)}^{\boxed{0< C_i \succ 2}}}{1 + \underbrace{\frac{2\beta^2\Delta t}{h^2}}_{2r}\theta\underbrace{\left(1-\cos\left(\frac{i\pi}{N}\right)\right)}_{\boxed{0< C_i \succ 2}}} \right| < 1 \quad \Bigg\} \quad -1 < \frac{1-2r(1-\theta)C_i}{1+2r\theta C_i} < 1$$

$$r := \frac{\beta^2\Delta t}{h^2}$$

$$C_i := \left(1-\cos\left(\frac{i\pi}{N}\right)\right)$$

Numerical solution of the heat equation
4.) Theta method (implicit) – stability analysis

$$\boxed{r := \frac{\beta^2 \Delta t}{h^2}}$$

1) Right hand side of the equation:

$$\boxed{-1 < \frac{1-2r(1-\theta)C_i}{\underbrace{1+2r\theta C_i}_{>0}}}$$

$$-1-2r\theta C_i < 1-2r(1-\theta)C_i \qquad /-1$$

$$-2-2r\theta C_i < -2rC_i + 2r\theta C_i \qquad /+2r\theta C_i$$

$$-2 < -2rC_i + 4r\theta C_i \qquad /:(-2)$$

$$1 > r(1-2\theta)C_i$$

$$\boxed{1-2\theta \leq 0 \quad (\theta \geq 0.5)}$$

$$1 > \underbrace{r(1-2\theta)}_{<0}C_i$$

unconditionally satisfied

$$\boxed{1-2\theta > 0 \quad (\theta < 0.5)}$$

$$\frac{1}{C_i(1-2\theta)} > r \qquad \frac{h^2}{\beta^2 \underbrace{C_i}_{\max 2} \underbrace{(1-2\theta)}_{\max 1}} > \Delta t \qquad \boxed{\frac{h^2}{2\beta^2(1-2\theta)} > \Delta t}$$

2) Left hand side of the equation:

$$\boxed{\frac{1-2r(1-\theta)C_i}{\underbrace{1+2r\theta C_i}_{>0}} < 1}$$

$$1-2r(1-\theta)C_i < 1+2r\theta C_i$$

$$2rC_i > 0$$

$$\Downarrow$$

unconditionally satisfied

Summary:
Instationary heat equation – what to solve?

- Method of lines

$$\mathbf{u}(t) = \sum_{j=1}^{N-1} \beta_j(t)\mathbf{v}_j \quad \beta_j(t) = \beta_j^0 e^{\lambda_j t} \quad \Rightarrow$$

find the eigenvalues (λ_j) and eigenvectors ($\mathbf{v}_j$) of matrix **A**

- Euler forward method

$$\mathbf{u}_{\mathrm{n+1}} = \underbrace{(\mathrm{I} + \Delta t\mathbf{A})}_{\mathbf{B}}\mathbf{u}_{\mathrm{n}} \quad \Rightarrow \quad \mathbf{u}_{\mathrm{n+1}} = \mathbf{B}\mathbf{u}_{\mathrm{n}}$$

- Euler backward method

$$\mathbf{u}_{\mathrm{n}} = \underbrace{(\mathrm{I} - \Delta t\mathbf{A})}_{\mathbf{B}_1}\mathbf{u}_{\mathrm{n+1}} \quad \Rightarrow \quad \mathbf{B}_1\mathbf{u}_{\mathrm{n+1}} = \mathbf{u}_{\mathrm{n}}$$

- Theta method

$$\underbrace{(\mathbf{I} - \theta\Delta t\mathbf{A})}_{\mathbf{B}_{1\theta}}\mathbf{u}_{\mathrm{n+1}} = \underbrace{(\mathbf{I} + (1-\theta)\Delta t\mathbf{A})}_{\mathbf{B}_{2\theta}}\mathbf{u}_{\mathrm{n}} \quad \Rightarrow \quad \mathbf{B}_{1\theta}\mathbf{u}_{\mathrm{n+1}} = \mathbf{B}_{2\theta}\mathbf{u}_{\mathrm{n}}$$

Solve system of equations for $\mathbf{u}_{\mathrm{n+1}}$

Summary:
Instationary heat equation – is it stable?

Stability checking from eigenvalue analysis:

- Method of lines

$$\mathbf{u}(t) = \sum_{j=1}^{N-1} \beta_j(t)\mathbf{v}_j \quad \beta_j(t) = \beta_j^0 e^{\lambda_j t} \Rightarrow \lambda_i(\boldsymbol{A}) \leq 0 \Rightarrow$$ uncond. stable

---

- Euler forward method

$$\mathbf{u}_{n+1} = \mathbf{B}\mathbf{u}_n$$

$$\nearrow \quad |\lambda_i(\boldsymbol{B})| \leq 1 \quad \Delta t < \frac{h^2}{2\beta^2} \Rightarrow$$ stable

$$\searrow \quad |\lambda_i(\boldsymbol{B})| > 1 \quad \Delta t \geq \frac{h^2}{2\beta^2} \Rightarrow$$ instable

---

- Euler backward method

$$\mathbf{B}_1\mathbf{u}_{n+1} = \mathbf{u}_n \qquad \left|\frac{1}{\lambda_i(\boldsymbol{B}_1)}\right| \leq 1 \Rightarrow$$ uncond. stable

---

- Theta method

$$\mathbf{B}_{1\theta}\mathbf{u}_{n+1} = \mathbf{B}_{2\theta}\mathbf{u}_n$$

$$\nearrow \quad \left|\frac{\lambda_i(\mathbf{B}_{2\theta})}{\lambda_i(\mathbf{B}_{1\theta})}\right| \leq 1$$

$\nearrow$ If $(\theta \geq 0.5)$

$\rightarrow$ If $(\theta < 0.5)$ and $\Delta t < \dfrac{h^2}{2\beta^2(1-2\theta)}$

$\Big\} \Rightarrow$ stable

$$\searrow \quad \frac{\lambda_i(\mathbf{B}_{2\theta})}{\lambda_i(\mathbf{B}_{1\theta})} < -1$$

If $(\theta < 0.5)$ and $\Delta t \geq \dfrac{h^2}{2\beta^2(1-2\theta)} \Rightarrow$ instable