Technische Universität Braunschweig

Introduction to PDEs and Numerical Methods

Lecture 5:

Analytical solution of ODEs and PDEs,
The Finite Difference Method

Dr. Noemi Friedman, 16.11.2016.

Overview of the course

- Introduction (definition of PDEs, classification, basic math, introductory examples of PDEs)

- Analytical solution of elementary PDEs (Fourier series/transform, separation of variables, Green's function)

- Numerical solutions of PDEs:

  - Finite difference method

  - Finite element method

Overview of this lecture

- Solving PDEs, analytical solution of ODEs
  - About existence and uniqueness of linear PDEs
  - Solution methods
    - Spectral method (Fourier analysis)
  - Essesntial ODEs
    - Solving homogenous second order ODEs
    - From homogenous to inhomogenous equation
    - Converting higher order ODEs to system of first order ODEs
    - Solving system of ODEs
- Introduction to the Finitie Difference Method (FDM)
  - Euler forward, Euler backward schemes and the theta-method
  - Approximation of higher derivatives
- Solving the heat equation with FDM
  - Semidiscretization- Method of Lines (spatial discretization)
  - Time discretization

Essential ODEs
solving linear systems↔analytical solution of linear ODEs

$\mathbf{Ax} = \mathbf{b}$

Existence:

$\mathbf{b} \in R(\mathbf{C})$ ($\mathbf{b}$ is in the range of $\mathbf{A}$)

Uniqueness:
let's suppose $\mathbf{y}$ and $\boldsymbol{z}$ are both solutions:

$\mathbf{Ay} = \mathbf{b}$ $\mathbf{Az} = \mathbf{b}$

$\mathbf{A}(\mathbf{y} - \mathbf{z}) = \mathbf{0}$ ⇒ if $\mathbf{y} \neq \boldsymbol{z}$ nontrivial solution

In other words, the nullspace of $\mathbf{A}$ is nontrivial.

The system has only unique solution if the nullspace of $\mathbf{A}$ is trivial, that is the only solution of
$\mathbf{Ax} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$

$Lu = f$ example: $L_D u = \frac{\partial^2 u}{\partial x^2}$

Existence:

$u \in R(L)$ (f is in the range of L)

Uniqueness:
let's suppose y and z are both solutions:

$Ly = f$ $Lz = f$

$L(y - z) = 0$ ⇒ if $y \neq z$ nontrivial solution

The system has only unique solution if the nullspace of L is trivial, that is the only solution of
$Lu = 0$ is the zero function

$\langle Ax, y\rangle = \langle x, A^T y\rangle$ for all $x, y \in \mathbb{R}^n$

Essential ODEs
solving linear systems↔analytical solution of linear ODEs

$$\mathbf{Ax} = \mathbf{b}$$

Solution:

If $N(\mathbf{A})$ is nontrivial, it has only solution if it satisfies a certain compatibility solution:
Adjoint operator: $\mathbf{A}^{\mathrm{T}}$

$$\mathbf{A}^{\mathrm{T}}\mathbf{w} = \mathbf{0} \Rightarrow \mathbf{w} \in N(\mathbf{A}^{\mathrm{T}})$$

$$\mathbf{w} \cdot \mathbf{b} = \mathbf{0}$$

If $N(\mathbf{A})$ is nontrivial, and if it has a solution, it has infinitely many:

$$\left.\begin{array}{l}\mathbf{Aw} = \mathbf{0} \\ \mathbf{Az} = \mathbf{b}\end{array}\right\} \mathbf{A}(\mathbf{z} + \alpha\mathbf{w}) = \mathbf{Az} + \alpha\mathbf{Aw} = \mathbf{b}$$

$\Rightarrow$ $\mathbf{z} + \alpha\mathbf{w}$ is also a solution

$$Lu = f$$

Solution:

If $N(\mathbf{A})$ is nontrivial, it has only solution if it satisfies a certain compatibility solution.

Adjoint operator L^*: $\langle Lu, v\rangle = \langle u, L^* v\rangle$

If $N(L)$ is nontrivial, and if it has a solution, it has infinitely many:

$$\left.\begin{array}{l}Lw = 0 \\ Lz = f\end{array}\right\} L(z + \alpha w) = Lz + \alpha Lw = f$$

$\Rightarrow$ $z + \alpha w$ is also a solution

Essential ODEs
solving linear systems↔analytical solution of linear ODEs

Uniqueness (example1):

$$L_D u = -\alpha \frac{\partial^2 u}{\partial x^2} \Rightarrow \left.\begin{aligned} -\alpha \frac{\partial^2 u(x)}{\partial x^2} &= f(x) \; x \in [0,l] \\ u(0) &= 0 \\ u(l) &= 0 \end{aligned}\right\} L_D : C_D^2[0,l] \to C[0,l]$$

The homogenous system:

$$-\alpha \frac{\partial^2 u(x)}{\partial x^2} = 0$$

$$\Rightarrow u(x) = ax + b \quad \left.\begin{aligned} u(0) = 0 &\Rightarrow b = 0 \\ u(l) = 0 &\Rightarrow a = 0 \end{aligned}\right\} \Rightarrow u(x) = 0 \Rightarrow$$

the trivial solution unique solution of $L_D u = f$

$$-\alpha \frac{d^2 u(x)}{dx^2} = f(x) \Rightarrow \alpha \frac{du(x)}{dx} = -\overbrace{\int_0^x f(s)ds}^{F(x)} + c_1 \Rightarrow \alpha u = -\int_0^x F(s)ds + c_1 x + c_2$$

$$u(0) = 0 \Rightarrow c_2 = 0$$

$$u(l) = 0 \Rightarrow c_1 = \frac{1}{l}\int_0^l \int_0^z f(s)dsdz$$

<u>Uniqueness (example2) :</u>

$L_N u = f \Rightarrow$

$$-\alpha u_{xx} = f(x) \quad x \in [0,l]$$
$$u_x(0) = 0$$
$$u_x(l) = 0$$

$\Big\}\ L_D: C_N^2[0,l] \to C[0,l]$

The homogenous system:

$$-\alpha \frac{\partial^2 u(x)}{\partial x^2} = 0$$

$\Rightarrow u(x) = ax + b \quad u_x(0) = 0 \Rightarrow a = 0, \quad u_x(l) = 0 \;\Big\} \Rightarrow u(x) = b \Rightarrow$ non trivial solution

if there is a solution, it is not unique

$$-\alpha \frac{d^2 u(x)}{dx^2} = f(x) \Rightarrow -\alpha \left[\frac{du(x)}{dx}\right]_0^l = \int_0^l f(x)dx \Rightarrow \int_0^l f(x)dx = 0$$

Essential ODEs
solving linear systems↔analytical solution of linear ODEs

$\mathbf{Ax} = \mathbf{b}$

Solution:

1) **General solution**

$$\mathbf{A}^{-1}\mathbf{Ax} = \mathbf{A}^{-1}\mathbf{b} \qquad \mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

2) **Direct solvers (Gauß elimination), iterative methods**

3) **Spectral method**

If $\mathbf{A}^{\mathrm{T}} = \mathbf{A}$ (real eigenvalues) $\quad \mathbf{A}\mathbf{v}_{\mathrm{i}} = \lambda_i \mathbf{v}_{\mathrm{i}}$

$$\mathbf{b} = \sum_i (\mathbf{v}_{\mathbf{i}} \cdot \mathbf{b})\mathbf{v}_{\mathrm{i}} \qquad \mathbf{x} = \sum_i (\mathbf{v}_{\mathbf{i}} \cdot \mathbf{x})\mathbf{v}_{\mathrm{i}} = \sum_i \alpha_i \mathbf{v}_{\mathrm{i}}$$

$$\mathbf{Ax} = \mathbf{b} \Rightarrow \mathbf{A}\sum_i \alpha_i \mathbf{v}_{\mathrm{i}} = \sum_i (\mathbf{v}_{\mathbf{i}} \cdot \mathbf{b})\mathbf{v}_{\mathrm{i}}$$

$$\sum_i \alpha_i \underbrace{\mathbf{A}\mathbf{v}_{\mathrm{i}}}_{\lambda_i \mathbf{v}_{\mathrm{i}}} = \sum_i (\mathbf{v}_{\mathbf{i}} \cdot \mathbf{b})\mathbf{v}_{\mathrm{i}} \Rightarrow \alpha_i \lambda_i = (\mathbf{v}_{\mathbf{i}} \cdot \mathbf{b})$$

$$\mathbf{x} = \sum_i \frac{(\mathbf{v}_{\mathbf{i}} \cdot \mathbf{b})}{\lambda_i}\mathbf{v}_{\mathrm{i}}$$

$Lu = f$

Solution:

1) **Direct integration**
Method of Green's functions

2) **Galerkin method/FD method**

3) **Fourier series** $\quad Lv_i = \lambda v_i$

$\langle Lu, v\rangle = \langle u, Lv\rangle$ (real eigenvalues)

$$f = \sum_i f_i v_i(x)$$

$$u = \sum_i u_i v_i(x)$$

solving linear systems↔analytical solution of linear ODEs

Solving ODEs with **Fourier series - example** $-\alpha \frac{d^2u}{dx^2} = f(x)$ $u(0) = 0$, $u(l) = 0$ $L_D u = f(x)$

1) Solve the eigenvalues-eigenfunctions $(\lambda_i, v_i(x))$

a) Can eigenfunctions form an orthogonal basis (is the operator symmetric)? $\langle Lu, v\rangle = \langle u, Lv\rangle$?

$$\langle Lu, v\rangle = -\alpha \int_0^l \frac{d^2u(x)}{dx^2} v(x)dx = \left[-\alpha \frac{du(x)}{dx} v(x)\right]_0^l + \alpha \int_0^l \frac{du(x)}{dx}\frac{dv(x)}{dx} dx$$

$$= \alpha \int_0^l \frac{du(x)}{dx}\frac{dv(x)}{dx} dx = \left[\alpha u(x) \frac{dv(x)}{dx}\right]_0^l - \alpha \int_0^l u(x) \frac{dv(x)}{dx^2} dx =$$

$$= \alpha \int_0^l u(x) \frac{dv(x)}{dx^2} dx = \langle u, Lv\rangle$$

b) Find eigenfunctions and eigenvalues $L_D v_i = \lambda v_i(x)$

$v_i = \sin\left(\frac{i\pi x}{l}\right)$ ➡ We try to find the solution in the form $u = \sum_i u_i \sin\left(\frac{i\pi x}{l}\right)$

Essential ODEs
solving linear systems↔analytical solution of linear ODEs

Solving ODEs with **Fourier series - example** $-\alpha \frac{d^2u}{dx^2} = f(x)$ $\quad u(0) = 0$, $u(l) = 0$ $\quad L_D u = f(x)$

2) Project $f(x)$ to the space spanned by the eigenfunctions:

$$f(x) = \sum_i f_i \sin\left(\frac{i\pi x}{l}\right) \qquad f_i = \frac{\left\langle f_i, \sin\left(\frac{i\pi x}{l}\right)\right\rangle}{\left\langle \sin\left(\frac{i\pi x}{l}\right), \sin\left(\frac{i\pi x}{l}\right)\right\rangle}$$

3) Solve the ODE for u_i:

$$-\alpha \frac{d^2}{dx^2}\sum_i u_i \sin\left(\frac{i\pi x}{l}\right) = -\alpha \sum_i u_i \frac{d^2}{dx^2}\sin\left(\frac{i\pi x}{l}\right) = \sum_i \alpha \frac{i^2\pi^2}{l^2} u_i \sin\left(\frac{i\pi x}{l}\right) = \sum_i f_i \sin\left(\frac{i\pi x}{l}\right)$$

$$\alpha \frac{i^2\pi^2}{l^2} u_i = f_i \;\Rightarrow\; u_i = \frac{l^2 f_i}{i^2\pi^2\alpha} \qquad u(x) = \sum_i u_i \sin\left(\frac{i\pi x}{l}\right)$$

Essential ODEs
Analytical solution of homogeneous second order ODEs

$$m\ddot{u} + c\dot{u} + ku = 0$$

Assuming the solution in the form:

$$u(\mathrm{t}) = e^{\sigma \mathrm{t}}$$

$$m\sigma^2 e^{\delta x} + c\sigma e^{\delta x} + k = 0$$

$$m\sigma^2 + c\sigma + k = 0 \quad (1)$$

$$\sigma_{1,2} = \frac{1}{2m}\left(-c \pm \sqrt{c^2 - 4mk}\right)$$

$$u = C_1 e^{\sigma_1 \mathrm{t}} + C_2 e^{\sigma_2 \mathrm{t}}$$

$$\frac{e^{(\alpha - i\beta)\mathrm{t}} + e^{(\alpha + i\beta)\mathrm{t}}}{2} u(\mathrm{t}) = e^{\alpha \mathrm{t}} \cos(\beta \mathrm{t})$$

$$\frac{e^{(\alpha - i\beta)\mathrm{t}} - e^{(\alpha + i\beta)\mathrm{t}}}{2} u(\mathrm{t}) = e^{\alpha \mathrm{t}} \sin(\beta \mathrm{t})$$

a) $c^2 - 4mk < 0$

$$\sigma_{1,2} = \alpha + i\beta$$

$$u(\mathrm{t}) = e^{(\alpha \pm i\beta)\mathrm{t}}$$

$$u(\mathrm{t}) = e^{\alpha \mathrm{t}}(\cos(\beta \mathrm{t}) \pm i \sin(\beta \mathrm{t}))$$

$$u(\mathrm{t}) = e^{\alpha \mathrm{t}}(\mathrm{C}_1 \cos(\beta \mathrm{t}) \pm C_2 \sin(\beta \mathrm{t}))$$

b) $c^2 - 4mk > 0$

$$\sigma_{1,2} = \pm\gamma \qquad \gamma \in \Re^-$$

$$u(\mathrm{t}) = C_1 e^{\gamma \mathrm{t}} + C_2 e^{-\gamma \mathrm{t}}$$

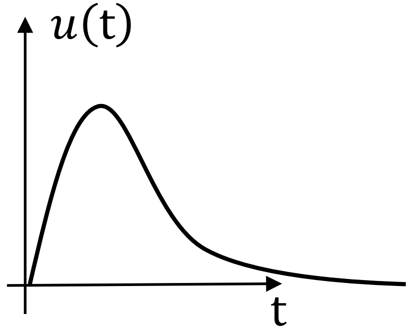
c) $c^2 - 4mk = 0$

$$\sigma_1 = \sigma_2 = \delta = \frac{-c}{2m}$$

$$u(\mathrm{t}) = C_1 e^{\delta \mathrm{t}} + C_2 \mathrm{t} e^{\delta \mathrm{t}} \qquad \gamma \in \Re$$

Essential ODEs
Analytical solution of homogeneous second order ODEs

$$m\ddot{u} + c\dot{u} + ku = 0$$

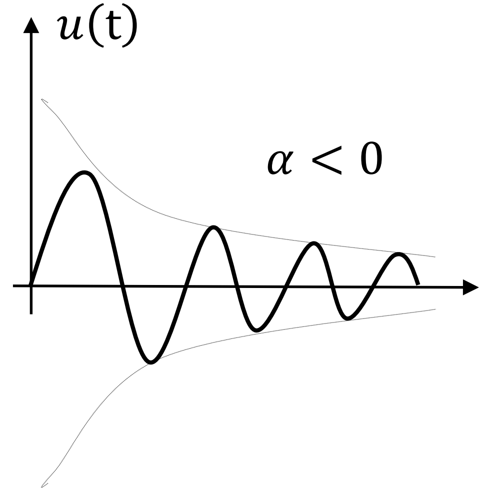


b) $c^2 - 4mk > 0$ $\quad \gamma \in \Re^-$

$$u(\mathrm{t}) = C_1 e^{\gamma \mathrm{t}} + C_2 e^{-\gamma \mathrm{t}}$$

a) $c^2 - 4mk < 0$

$$u(\mathrm{t}) = e^{\alpha \mathrm{t}}(\mathrm{C}_1 \cos(\beta \mathrm{t}) \pm C_2 \sin(\beta \mathrm{t}))$$



c) $c^2 - 4mk = 0$

$$u(\mathrm{t}) = C_1 e^{\delta \mathrm{t}} + C_2 \mathrm{t} e^{\delta \mathrm{t}} \qquad \gamma \in \Re$$

$$m\ddot{u}_h + c\dot{u}_h + ku_h = 0 \Rightarrow u_h = c_1 u_{1h} + c_2 u_{2h}$$ general solution of the homogenous equation

$$m\ddot{u} + c\dot{u} + ku = f \Rightarrow u_1, \; u_2$$ two particular solutions of the inhomogenous equation

Theorem:

$u_1 - u_2$ is a solution to the homogenous equation $\Rightarrow$ $u_1 - u_2 = c_1 u_{1h} + c_2 u_{2h}$

$$u_1 = c_1 u_{1h} + c_2 u_{2h} + u_2 = u_h + u_2$$

Proof:

$$m\frac{\partial^2}{\partial t^2}(u_1 - u_2) + c\frac{\partial}{\partial t}(u_1 - u_2) + k(u_1 - u_2) = 0$$

$$\underbrace{\left(m\frac{\partial^2}{\partial t^2}u_1 + c\frac{\partial}{\partial t}u_1 + ku_1\right)}_{f} - \underbrace{\left(m\frac{\partial^2}{\partial t^2}u_2 + c\frac{\partial}{\partial t}u_2 + ku_2\right)}_{f} = 0$$

The general solution of the inhomogenous equation can be written as the sum of the general solution of the homogenous eq. and one particular solution of the inhomogenous eq.

Essential ODEs
Converting higher order ODEs to first order system

$$m\ddot{u} + c\dot{u} + ku = f$$

$$u_1 := u$$

$$u_2 := \dot{u} \quad \Rightarrow \quad \dot{u}_1 = u_2$$

$$m\dot{u}_2 + cu_2 + ku_1 = f \quad \Rightarrow \quad \dot{u}_2 = -\frac{c}{m}u_2 - \frac{k}{m}u_1 + \frac{f}{m}$$

$$\frac{\partial}{\partial t}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{f}{m} \end{bmatrix}$$

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}$$

$$\mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

$$\boldsymbol{f}(t) = \begin{bmatrix} 0 \\ \frac{f(t)}{m} \end{bmatrix}$$

$$\frac{\partial}{\partial t}\mathbf{u}(t) = \boldsymbol{A}\mathbf{u}(t) + \boldsymbol{f}(t)$$

Essential ODEs
Converting higher order ODEs to first order system

$$\frac{\partial}{\partial t}\mathbf{u}(t) = \boldsymbol{A}\mathbf{u}(t) + \boldsymbol{f}(t)$$

Let's solve first the homogenous equation

$$\frac{\partial}{\partial t}\mathbf{u}(t) = \boldsymbol{A}\mathbf{u}(t)$$

Let's suppose that the solution has a form:

$$\mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} := \alpha(t)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \alpha(t)\boldsymbol{v}$$

$$\frac{\partial}{\partial t}\alpha(t)\boldsymbol{v} = \alpha(t)\boldsymbol{A}\boldsymbol{v}$$

$$\boldsymbol{A}\boldsymbol{v} = \lambda\boldsymbol{v}$$

$\boldsymbol{v}$ is an eigenvector of $\boldsymbol{A}$

$$\frac{\partial}{\partial t}\alpha(t)\boldsymbol{v} = \alpha(t)\lambda\boldsymbol{v}$$

$$\frac{\partial}{\partial t}\alpha(t) = \lambda\alpha(t)$$

$$\alpha_i(t) = C_i e^{\lambda_i(t-t_0)}$$

$$u_0 = u(0) = C_i\boldsymbol{v}_i$$

$$\alpha_i(0) = C_i$$

$$\mathbf{u}_\mathbf{i}(t) = C_i e^{\lambda_i(t-t_0)}\boldsymbol{v}_{\boldsymbol{i}}$$

$$\boldsymbol{u}(t) = \sum_i C_i e^{\lambda_i(t-t_0)}\boldsymbol{v}_{\boldsymbol{i}}$$

Overview of second part

- Essential ODEs

- The heat equation
  - Analytical solution
  - Semidiscretization- Method of Lines (spatial discretization)
  - Time discretization

Numerical solution of the heat equation
Finite Difference Method – derivation of difference operators

Derivation of $u'(x)$

$$u(x+h) = u(x) + u'(x)h + \frac{1}{2}u''(x)h^2 + O(h^3) \qquad (1)$$

$$u(x-h) = u(x) - u'(x)h + \frac{1}{2}u''(x)h^2 + O(h^3) \qquad (2)$$

Subtracting from eq. (1) eq. (2) results in:

$$u(x+h) - u(x-h) = 2u'(x)h + O(h^3)$$

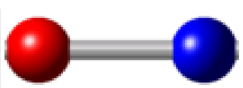
$$u'(x) = \frac{u(x+h) - u(x-h)}{2h} + O(h^2)$$

$$u'_k = \frac{u_{k+1} - u_{k-1}}{2h} + O(h^2)$$

From only (1) (EULER FORWARD)

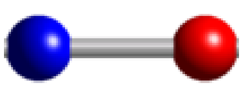
$$u'(x) = \frac{u(x+h) - u(x)}{h} + O(h)$$

$$u'_k = \frac{u_{k+1} - u_k}{h} + O(h)$$



From only (2) EULER BACKWARD

$$u'(x) = \frac{u(x) - u(x-h)}{h} + O(h)$$



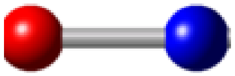
$$u'_{k+1} = \frac{u_{k+1} - u_k}{h} + O(h)$$

Numerical solution of the heat equation
Finite Difference Method – derivation of difference operators

Example for using the **two point stencil**

Forward differences – explicit method

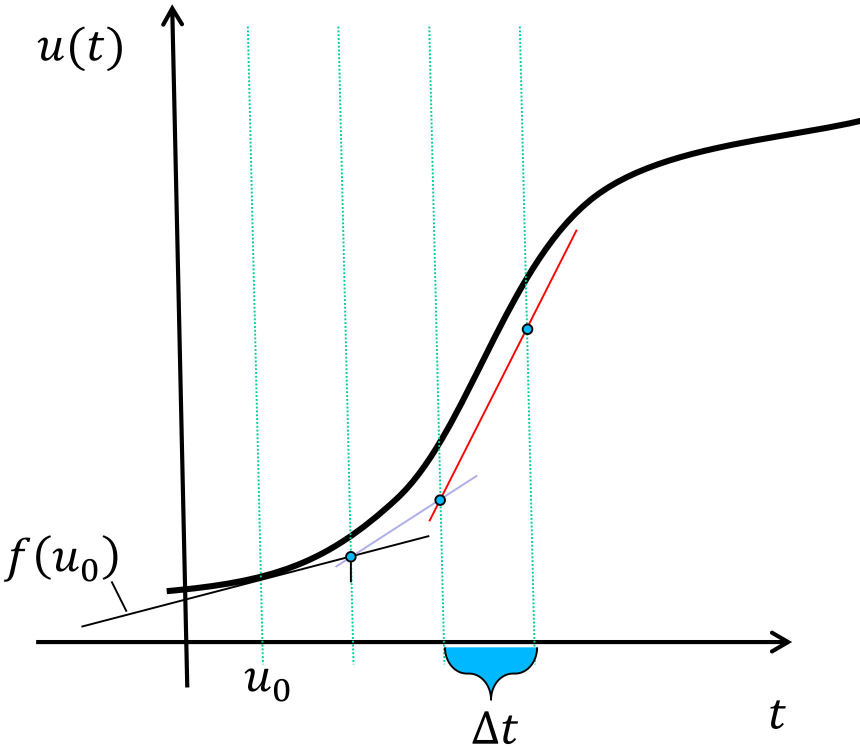
$$u'_k = \frac{u_{k+1} - u_k}{\Delta t} + O(\Delta t)$$



$$u'(t) = f\big(u(t)\big) \qquad B.C.: u(0) = \bar{u} = u_0$$

$$u'_k = \frac{u_{k+1} - u_k}{\Delta t} = f(u_k)$$

$$u_{k+1} = u_k + \Delta t f(u_k)$$

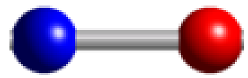


Numerical solution of the heat equation
Finite Difference Method – derivation of difference operators

Example for using the **two point stencil**

Backward differences – implicit method

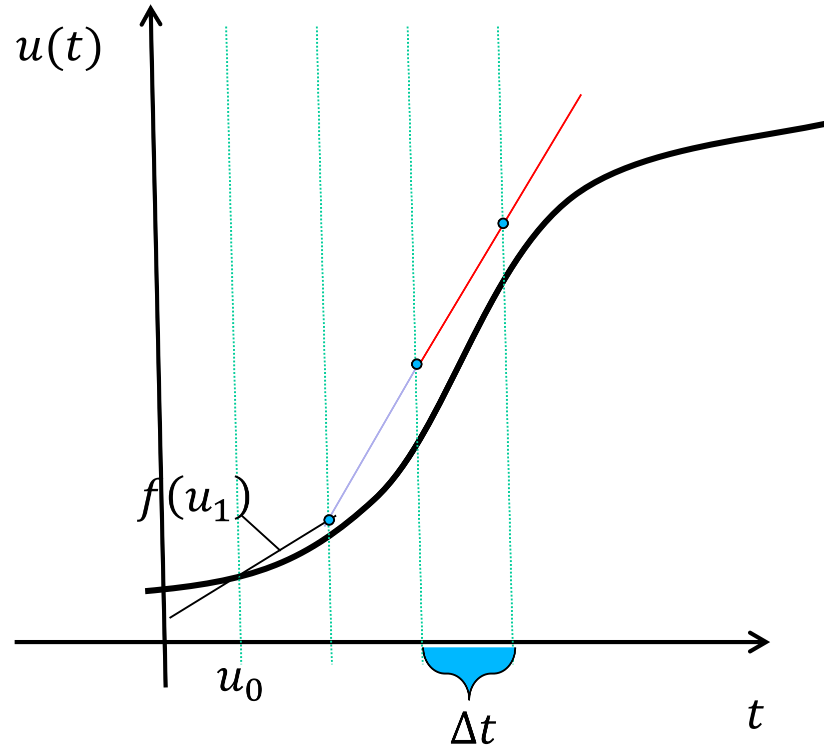
$$u'_{k+1} = \frac{u_{k+1} - u_k}{\Delta t} + O(h)$$



$$u'(t) = f\big(u(t)\big) \qquad B.C.: u(0) = \bar{u} = u_0$$

$$u'_{k+1} = \frac{u_{k+1} - u_k}{\Delta t} = f(u_{k+1})$$

$$u_{k+1} = \Delta t f(u_{k+1}) + u_k$$



Theta method

$$u'_{k} = \frac{u_{k+1} - u_k}{\Delta t} + O(h)$$

$$u'_{k+1} = \frac{u_{k+1} - u_k}{\Delta t} + O(h)$$

$$u'_{k+\theta} = \frac{u_{k+1} - u_k}{\Delta t} = \theta f(u_{k+1}) + (1-\theta) f(u_k)$$