Technische
Universität
Braunschweig

Introduction to PDEs and Numerical Methods

Lecture 5:

Analytical solution of ODEs and PDEs,
The Finite Difference Method

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Overview of the course

- Introduction (definition of PDEs, classification, basic math, introductory examples of PDEs)
- Analytical solution of elementary PDEs (Fourier series/transform, separation of variables, Green's function)
- Numerical solutions of PDEs:
  - **Finite difference method**
  - Finite element method

Overview of this lecture

- Solving PDEs, analytical solution of ODEs
  - About existence and uniqueness of linear PDEs
  - Solution methods
    - Spectral method (Fourier analysis)
  - Essential ODEs
    - Solving homogenous second order ODEs
    - From homogenous to inhomogenous equation
    - Converting higher order ODEs to system of first order ODEs
    - Solving system of ODEs
- Introduction to the Finitie Difference Method (FDM)
  - Euler forward, Euler backward schemes and the theta-method
  - Approximation of higher derivatives
- Solving the heat equation with FDM
  - Semidiscretization- Method of Lines (spatial discretization)
  - Time discretization

Analytical solution of homogeneous second order ODEs

$$m\ddot{u} + c\dot{u} + ku = 0$$

Assuming the solution in the form:

$$u(\mathrm{t}) = e^{\sigma \mathrm{t}}$$

$$m\sigma^2 e^{\sigma x} + c\sigma e^{\sigma x} + k = 0$$

$$m\sigma^2 + c\sigma + k = 0 \quad (1)$$

$$\sigma_{1,2} = \frac{1}{2m}\left(-c \pm \sqrt{c^2 - 4mk}\right)$$

$$u = C_1 e^{\sigma_1 \mathrm{t}} + C_2 e^{\sigma_2 \mathrm{t}}$$

$$\frac{e^{(\alpha - i\beta)\mathrm{t}} + e^{(\alpha + i\beta)\mathrm{t}}}{2} u(\mathrm{t}) = e^{\alpha t}\cos(\beta \mathrm{t})$$

$$\frac{e^{(\alpha - i\beta)\mathrm{t}} - e^{(\alpha + i\beta)\mathrm{t}}}{2} u(\mathrm{t}) = e^{\alpha t}\sin(\beta \mathrm{t})$$

a) $c^2 - 4mk < 0$

$$\sigma_{1,2} = \alpha + i\beta$$

$$u(\mathrm{t}) = e^{(\alpha \pm i\beta)\mathrm{t}}$$

$$u(\mathrm{t}) = e^{\alpha t}(\cos(\beta \mathrm{t}) \pm i\sin(\beta \mathrm{t}))$$

$$u(\mathrm{t}) = e^{\alpha t}(\mathrm{C}_1\cos(\beta \mathrm{t}) \pm C_2\sin(\beta \mathrm{t}))$$

b) $c^2 - 4mk > 0$

$$\sigma_{1,2} = \pm\gamma \qquad \gamma \in \Re^-$$

$$u(\mathrm{t}) = C_1 e^{\gamma \mathrm{t}} + C_2 e^{-\gamma \mathrm{t}}$$

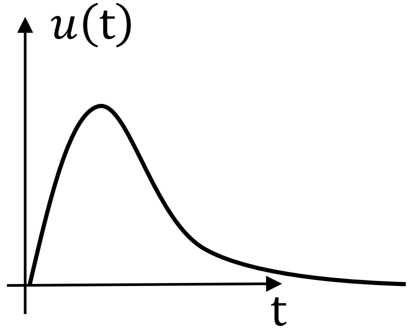
c) $c^2 - 4mk = 0$

$$\sigma_1 = \sigma_2 = \delta = \frac{-c}{2m}$$

$$u(\mathrm{t}) = C_1 e^{\delta \mathrm{t}} + C_2 \mathrm{t} e^{\delta \mathrm{t}} \qquad \gamma \in \Re$$

Essential ODEs
Analytical solution of homogeneous second order ODEs

$$m\ddot{u} + c\dot{u} + ku = 0$$

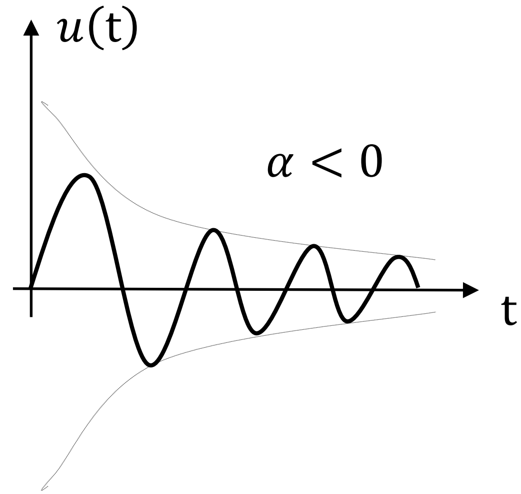


b) $c^2 - 4mk > 0$ $\gamma \in \Re^-$

$$u(\mathrm{t}) = C_1 e^{\gamma \mathrm{t}} + C_2 e^{-\gamma \mathrm{t}}$$

a) $c^2 - 4mk < 0$

$$u(\mathrm{t}) = e^{\alpha \mathrm{t}}(C_1 \cos(\beta \mathrm{t}) \pm C_2 \sin(\beta \mathrm{t}))$$



c) $c^2 - 4mk = 0$

$$u(\mathrm{t}) = C_1 e^{\delta \mathrm{t}} + C_2 \mathrm{t} e^{\delta \mathrm{t}} \qquad \gamma \in \Re$$

Essential ODEs
Solving inhomogenous linear ODEs

$$m\ddot{u}_h + c\dot{u}_h + ku_h = 0 \quad \Rightarrow \quad u_h = c_1 u_{1h} + c_2 u_{2h}$$

general solution of the homogenous equation

$$m\ddot{u} + c\dot{u} + ku = f \quad \Rightarrow \quad u_1, \; u_2$$

two particular solutions of the inhomogenous equation

Theorem:

$u_1 - u_2$ is a solution to the homogenous equation $\Rightarrow$ $u_1 - u_2 = c_1 u_{1h} + c_2 u_{2h}$

$$u_1 = c_1 u_{1h} + c_2 u_{2h} + u_2 = u_h + u_2$$

The general solution of the inhomogenous equation can be written as the sum of the general solution of the homogenous eq. and one particular solution of the inhomogenous eq.

Proof:

$$m\frac{\partial^2}{\partial t^2}(u_1 - u_2) + c\frac{\partial}{\partial t}(u_1 - u_2) + k(u_1 - u_2) = 0$$

$$\underbrace{\left(m\frac{\partial^2}{\partial t^2}u_1 + c\frac{\partial}{\partial t}u_1 + ku_1\right)}_{f} - \underbrace{\left(m\frac{\partial^2}{\partial t^2}u_2 + c\frac{\partial}{\partial t}u_2 + ku_2\right)}_{f} = 0$$

Converting higher order ODEs to first order system

$$m\ddot{u} + c\dot{u} + ku = f$$

$$u_1 := u$$

$$u_2 := \dot{u} \quad \Rightarrow \quad \dot{u}_1 = u_2$$

$$m\dot{u}_2 + cu_2 + ku_1 = f \quad \Rightarrow \quad \dot{u}_2 = -\frac{c}{m}u_2 - \frac{k}{m}u_1 + \frac{f}{m}$$

$$\frac{\partial}{\partial t}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{f}{m} \end{bmatrix}$$

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}$$

$$\mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

$$\boldsymbol{f}(t) = \begin{bmatrix} 0 \\ \frac{f(t)}{m} \end{bmatrix}$$

$$\frac{\partial}{\partial t}\mathbf{u}(t) = \boldsymbol{A}\mathbf{u}(t) + \boldsymbol{f}(t)$$

Converting higher order ODEs to first order system

$$\frac{\partial}{\partial t}\mathbf{u}(t) = \boldsymbol{A}\mathbf{u}(t) + \boldsymbol{f}(t)$$

Let's solve first the homogenous equation

$$\frac{\partial}{\partial t}\mathbf{u}(t) = \boldsymbol{A}\mathbf{u}(t)$$

Let's suppose that the solution has a form:

$$\mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} := \alpha(t)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \alpha(t)\boldsymbol{v}$$

$$\Rightarrow \frac{\partial}{\partial t}\alpha(t)\boldsymbol{v} = \alpha(t)\boldsymbol{A}\boldsymbol{v}$$

$$\boldsymbol{A}\boldsymbol{v} = \lambda\boldsymbol{v}$$

$\boldsymbol{v}$ is an eigenvector of $\boldsymbol{A}$

$$\frac{\partial}{\partial t}\alpha(t)\boldsymbol{v} = \alpha(t)\lambda\boldsymbol{v}$$

$$\frac{\partial}{\partial t}\alpha(t) = \lambda\alpha(t)$$

$$\alpha_i(t) = C_i e^{\lambda_i(t-t_0)}$$

$$u_0 = u(0) = C_i\boldsymbol{v}_i$$

$$\alpha_i(0) = C_i$$

$$\mathbf{u_i}(t) = C_i e^{\lambda_i(t-t_0)}\boldsymbol{v_i}$$

$$\boldsymbol{u}(t) = \sum_i C_i e^{\lambda_i(t-t_0)}\boldsymbol{v_i}$$

II. The Finite Difference Method

- Solving PDEs with the Finite Difference method

Numerical solution of the heat equation
Finite Difference Method – derivation of difference operators

Derivation of $u'(x)$

$$u(x+h) = u(x) + u'(x)h + \frac{1}{2}u''(x)h^2 + O(h^3) \qquad (1)$$

$$u(x-h) = u(x) - u'(x)h + \frac{1}{2}u''(x)h^2 + O(h^3) \qquad (2)$$

Subtracting from eq. (1) eq. (2) results in:

$$u(x+h) - u(x-h) = 2u'(x)h + O(h^3)$$

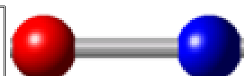
$$u'(x) = \frac{u(x+h) - u(x-h)}{2h} + O(h^2)$$

$$u'_k = \frac{u_{k+1} - u_{k-1}}{2h} + O(h^2)$$

From only (1) (EULER FORWARD)

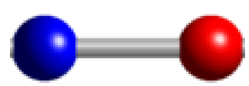
$$u'(x) = \frac{u(x+h) - u(x)}{h} + O(h)$$

$$u'_k = \frac{u_{k+1} - u_k}{h} + O(h)$$



From only (2) EULER BACKWARD

$$u'(x) = \frac{u(x) - u(x-h)}{h} + O(h)$$



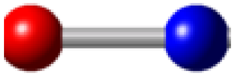
$$u'_{k+1} = \frac{u_{k+1} - u_k}{h} + O(h)$$

Numerical solution of the heat equation
Finite Difference Method – derivation of difference operators

Example for using the **two point stencil**

Forward differences – explicit method

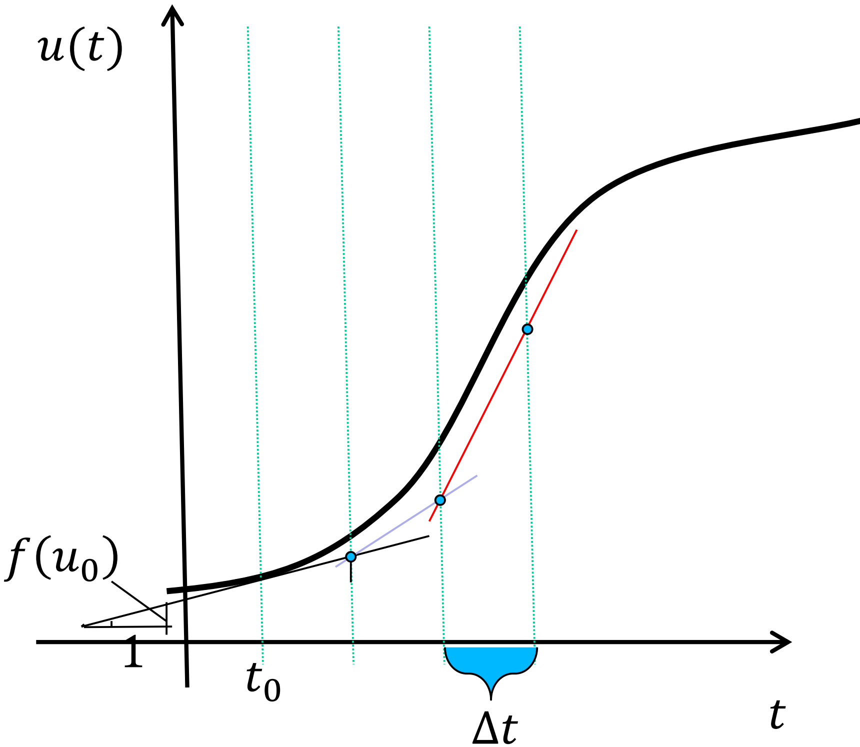
$$u'_k = \frac{u_{k+1} - u_k}{\Delta t} + O(\Delta t)$$



$$u'(t) = f\big(u(t)\big) \qquad B.C.: u(0) = \bar{u} = u_0$$

$$u'_k = \frac{u_{k+1} - u_k}{\Delta t} = f(u_k)$$

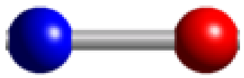
$$u_{k+1} = u_k + \Delta t f(u_k)$$



Example for using the **two point stencil**

Backward differences – implicit method

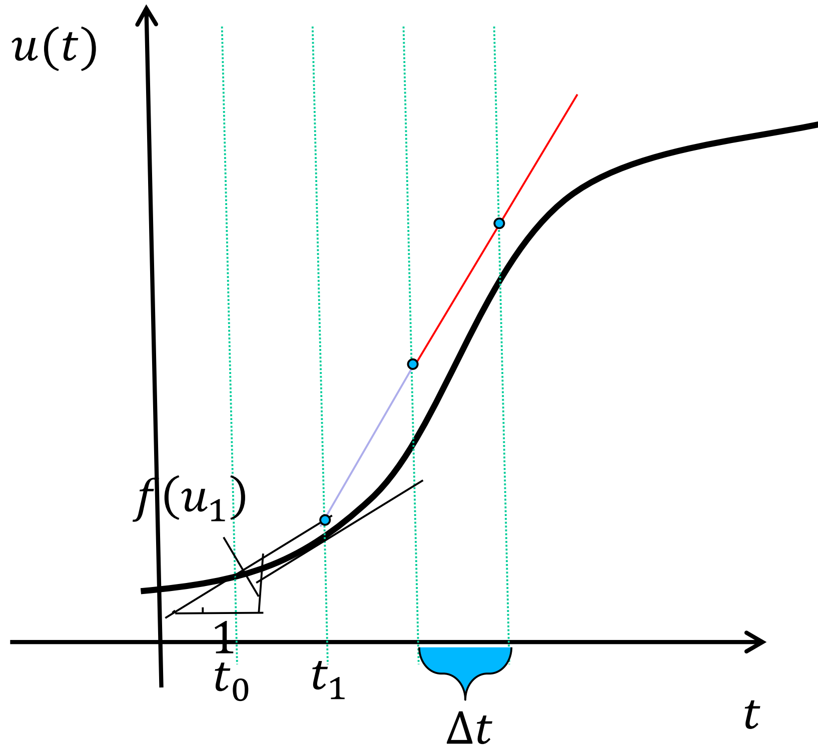
$$u'_{k+1} = \frac{u_{k+1} - u_k}{\Delta t} + O(h)$$



$$u'(t) = f\big(u(t)\big) \qquad B.C.: u(0) = \bar{u} = u_0$$

$$u'_{k+1} = \frac{u_{k+1} - u_k}{\Delta t} = f(u_{k+1})$$

$$u_{k+1} = \Delta t f(u_{k+1}) + u_k$$

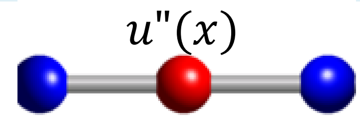


Theta method

$$u'_k = \frac{u_{k+1} - u_k}{\Delta t} + O(h)$$

$$u'_{k+1} = \frac{u_{k+1} - u_k}{\Delta t} + O(h)$$

$$u'_{k+\theta} = \frac{u_{k+1} - u_k}{\Delta t} = \theta f(u_{k+1}) + (1-\theta) f(u_k)$$



Derivation of $u''(x)$ – **the three point stencil**

$$u(x+h) = u(x) + u'(x)h + \frac{1}{2}u''(x)h^2 + \frac{1}{3!}u'''(x)h^3 + \frac{1}{4!}u^{IV}(x)h^4 + \frac{1}{5!}u^{V}(x)h^5 + O(h^6) \quad (3)$$

$$u(x-h) = u(x) - u'(x)h + \frac{1}{2}u''(x)h^2 - \frac{1}{3!}u'''(x)h^3 + \frac{1}{4!}u^{IV}(x)h^4 - \frac{1}{5!}u^{V}(x)h^5 + O(h^6) \quad (4)$$

Adding eq. (3) and eq. (4) results in:

$$u(x+h) + u(x-h) = 2u(x) + 0 + u''(x)h^2 + 0 + \frac{2}{4!}u^{IV}(x)h^4 + 0 + O(h^6)$$

$$u''(x) = \frac{u(x+h) - 2u(x) + u(x-h)}{h^2} + O(h^2)$$

truncation error:

$$\frac{1}{12}\frac{d^4u}{dx^4}h^2 + O(h^4)$$

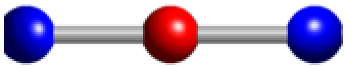
truncation constant

$$u''_j = \frac{u_{j+1} - 2u_j + u_{j-1}}{h^2} + O(h^2)$$

Numerical solution of the instationary heat equation
Method of lines (semi-discretized heat equation)

$$\frac{\partial u(x,t)}{\partial t} - \beta^2 \frac{\partial^2 u}{\partial x^2} = 0 \quad \Rightarrow \quad \frac{\partial u(x,t)}{\partial t} = \beta^2 \frac{\partial^2 u}{\partial x^2}$$

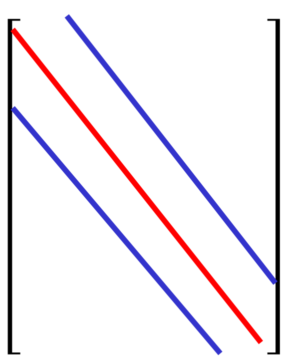
Approximate second derivate with the three point stencil (spatial discretisation of the heat eq.)



$$\frac{\partial^2 u_j}{\partial x^2} = \frac{1}{h^2}(u_{j+1} - 2u_j + u_{j-1}) + O(h^2)$$

$$\frac{\partial u_j(t)}{\partial t} = \frac{\beta^2}{h^2}\left(u_{j+1} - 2u_j + u_{j-1}\right) + O(h^2)$$

$$\frac{d}{dt}\begin{bmatrix} u_1(t) \\ \vdots \\ u_{j-1}(t) \\ u_j(t) \\ u_{j+1}(t) \\ \vdots \\ u_{N-1}(t) \end{bmatrix} = \frac{-\beta^2}{h^2}$$



$$\begin{bmatrix} u_1(t)) + \frac{\beta^2}{h^2} u_0(t) \\ \vdots \\ u_{j-1}(t) \\ u_j(t) \\ u_{j+1}(t) \\ \vdots \\ u_{N-1}(t)) + \frac{\beta^2}{h^2} u_0(t) \end{bmatrix}$$

$$\frac{d}{dt}\mathbf{u}(t) = \mathbf{A}\mathbf{u}(t)$$

$$\begin{bmatrix} 2 & -1 & 0 & & 0 \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ 0 & & & -1 & 2 \end{bmatrix}$$

Numerical solution of the instationary heat equation
Method of lines (semi-discretized heat equation)

$$\frac{\partial}{\partial t}\boldsymbol{u} = \frac{\beta^2}{h^2}\left(u_{j+1} - 2u_j + u_{j-1}\right) + O\left(h^2\right) = f_j$$

truncation error:

$$\frac{1}{12}\frac{d^4u}{dx^4}h^2 + O(h^4)$$

truncation constant

$$\frac{\partial}{\partial t}\boldsymbol{u} = \frac{-\beta^2}{h^2}\begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & & & \\ & & & 2 & -1 & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{bmatrix}\begin{bmatrix} u_1(t) \\ \vdots \\ u_{j-1}(t) \\ u_j(t) \\ u_{j+1}(t) \\ \vdots \\ u_{N-1}(t) \end{bmatrix} + \begin{bmatrix} f_1(t) + \frac{\beta^2}{h^2}u_0(t) \\ \vdots \\ f_{j-1}(t) \\ f_j(t) \\ f_{j+1}(t) \\ \vdots \\ f_{N-1}(t) + \frac{\beta^2}{h^2}u_N(t) \end{bmatrix}$$

$$\frac{\partial}{\partial t}\boldsymbol{u} = \boldsymbol{A}\boldsymbol{u}(t) + \boldsymbol{f}(t)$$

Numerical solution of the instationary heat equation
Method of lines (semi-discretized heat equation)

Solve analyticaly the system of ODEs:

$$\frac{d}{dt}\boldsymbol{u} = \boldsymbol{A}\boldsymbol{u}(t) + \boldsymbol{f}(t)$$

(see slide: 15)

$$a := \frac{-2\beta^2}{h^2} \qquad b := \frac{\beta^2}{h^2} \qquad \boldsymbol{A} = \begin{bmatrix} a & b & & & & \\ b & a & b & & & \\ & b & a & & & \\ & & & a & b & \\ & & & b & a & b \\ & & & & b & a \end{bmatrix}$$

1.) Find eigenvalues (λ_i) and eigenvectors ($\boldsymbol{v_i}$) of $\boldsymbol{A}$:

$$\lambda_i = a + 2b\cos\left(\frac{i\pi}{N}\right) = \frac{2\beta^2}{h^2}\left(\cos\left(\frac{i\pi}{N}\right) - 1\right) \qquad \boldsymbol{v_i} = \begin{bmatrix} v_i^1 \\ \vdots \\ v_i^{N-1} \end{bmatrix} \qquad v_i^k = \sin\left(\frac{ik\pi}{N}\right)$$

$$i, k = 1..N-1$$

2.) Write initial condition in the basis of the eigenvectors: $\boldsymbol{u}(0) = C_i \boldsymbol{v}_i$

3.) Look for the solution in the form: $\boldsymbol{u}(t) = \sum_i \alpha_i(t)\boldsymbol{v_i} \qquad \alpha_i(0) = C_i$

The solution to the homogenous equations: $\alpha_i(t) = C_i e^{\lambda_i(t-t_0)}$ $\mathbf{u_{ih}}(t) = C_i e^{\lambda_i(t-t_0)}\boldsymbol{v_i}$

$$\mathbf{u_h}(t) = \sum_i c_i e^{\lambda_i(t-t_0)}\boldsymbol{v_i}$$

$$\lambda_i = \underbrace{\frac{2\beta^2}{h^2}}_{>0}\Bigg(\underbrace{\cos\left(\frac{i\pi}{N}\right)}_{<1} - 1\Bigg)$$

Numerical solution of the instationary heat equation
Method of lines (semi-discretized heat equation)

$$\frac{d}{dt}\boldsymbol{u} = \boldsymbol{A}\boldsymbol{u}(t) + \boldsymbol{f}(t)$$

$$\mathbf{u_h}(t) = \sum_i c_i e^{\lambda_i (t-t_0)} \boldsymbol{v_i}$$

4.) Solve one particular solution of the inhomogenous system of ODEs:

4.1.) Write r.h.t in the basis of the eigenvectors : $\boldsymbol{f}(t) = \sum_i f_i \boldsymbol{v_i}$

4.2.) Look for the solution in the form: $\boldsymbol{u}(t) = \sum_i u_i(t) \boldsymbol{v_i}$

$$\frac{d}{dt}\boldsymbol{u} - \boldsymbol{A}\boldsymbol{u}(t) = \frac{d}{dt}\sum_i u_i(t)\boldsymbol{v_i} - \boldsymbol{A}\sum_i u_i(t)\boldsymbol{v_i} = \sum_i \frac{d}{dt}u_i(t)\boldsymbol{v_i} - \sum_i u_i(t)\boldsymbol{A}\boldsymbol{v_i}$$

$$= \sum_i \left(\frac{d}{dt}u_i(t) - u_i(t)\lambda_i\right)\boldsymbol{v_i}$$

5.)

$$\sum_i \left(\frac{d}{dt}u_i(t) - u_i(t)\lambda_i\right)\boldsymbol{v_i} = \sum_i f_i \boldsymbol{v_i} \implies \left(\frac{d}{dt}u_i(t) - u_i(t)\lambda_i\right) = f_i$$

FD discretization of the Poisson equation-hom. N. BC

$$\frac{-1}{h^2}\left(u_{j+1} - 2u_j + u_{j-1}\right) + O(h^2) = f_j$$

truncation error:

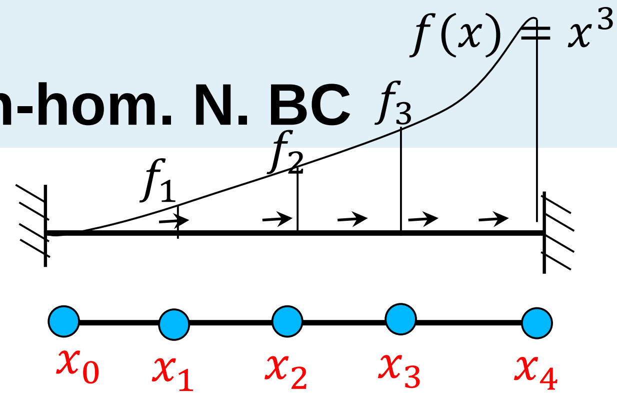
$$\frac{1}{12}\frac{d^4u}{dx^4}h^2 + O(h^4)$$

truncation constant

$$\frac{1}{h^2}\begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & & & \\ & & & 2 & -1 & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{bmatrix}\begin{bmatrix} u_1 \\ \vdots \\ u_{j-1} \\ u_j \\ u_{j+1} \\ \vdots \\ u_{N-1} \end{bmatrix} = \begin{bmatrix} f_1 \\ \vdots \\ f_{j-1} \\ f_j \\ f_{j+1} \\ \vdots \\ f_{N-1} \end{bmatrix}$$

FD discretization of the Poisson equation-hom. N. BC

$$\frac{-1}{h^2}\left(u_{j+1} - 2u_j + u_{j-1}\right) + O(h^2) = f_j$$



Exercise 1: *FD approximation of the Poisson equation with homogenous Dirichlet B.C.s*

Consider the boundary value problem

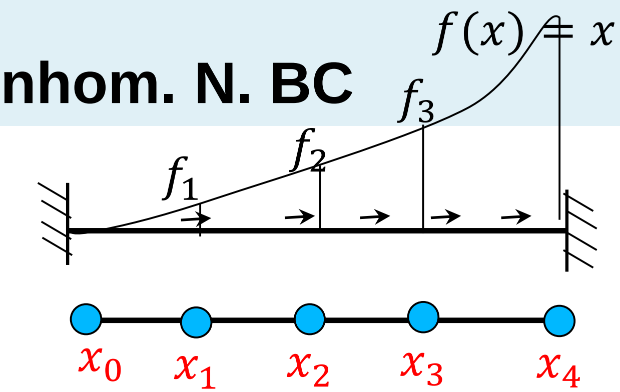
$$-u''(x) = x^3 \quad u(0) = 0 \quad u(1) = 0,$$

$$\frac{1}{h^2}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

$$\frac{1}{0.25^2}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0.25^3 \\ 0.5^3 \\ 0.75^3 \end{bmatrix}$$

FD discretization of the Poisson equation-inhom. N. BC

$$\frac{-1}{h^2}\left(u_{j+1} - 2u_j + u_{j-1}\right) + O(h^2) = f_j$$



Exercise 2: *FD approximation of the Poisson equation with inhomogenous Dirichlet B.C.s*

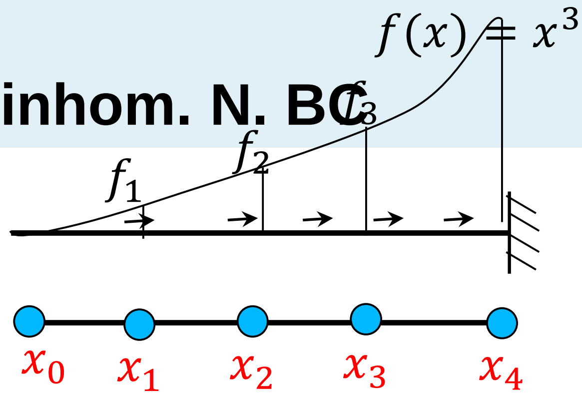
Consider the boundary value problem

$$-u''(x) = x^3 \quad u(0) = 1 \quad u(1) = 2,$$

$$\frac{1}{h^2}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} - \frac{1}{h^2}\begin{bmatrix} u_0 \\ 0 \\ u_4 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

$$\frac{1}{h^2}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} f_1 + \frac{1}{h^2}u(0) \\ f_2 \\ f_3 + \frac{1}{h^2}u(1) \end{bmatrix} \qquad \frac{1}{0.25^2}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0.25^3 + 1 \\ 0.5^3 \\ 0.75^3 + 2 \end{bmatrix}$$

FD discretization of the Poisson equation-inhom. N. BC



$$\frac{-1}{h^2}\left(u_{j+1} - 2u_j + u_{j-1}\right) + O(h^2) = f_j$$

Exercise 3: *FD approximation of the Poisson equation with mixed B.C.s*

Consider the boundary value problem

$$-u''(x) = x^3 \quad u'(0) = 0 \quad u(1) = 0,$$

The first derivative in the Neumann B.C. can be approximated by:

$$u'_n = \frac{u_{n+1} - u_n}{h}$$

$$\frac{1}{h^2}\begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix}$$