Introduction to PDEs and Numerical Methods

Lecture 4:

Analytical solution of ODEs and PDEs

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Overview of the course

- Introduction (definition of PDEs, classification, basic math, introductory examples of PDEs)

- Analytical solution of elementary PDEs (Fourier series/transform, separation of variables, Green's function)

- Numerical solutions of PDEs:

  - Finite difference method

  - Finite element method

Overview of this lecture

The Fourier-series and Fourier-transform

- On the choice of inner product for real and complex valued functions
- Fourier-series of periodic functions with period 1
- Connection between series with $\sin$ and $\cos$ and the Fourier-series
- Properties of the Fourier-series (e.g. for even and odd functions), derivatives, …
- Fourier-series of periodic functions with different period
- Examples

Chose inner product by preserving validity of Pithagoream-theorem in the real valued function space

The Pithagorean-theorem: $\|f+g\|^2 = \|f\|^2 + \|g\|^2$

The Pithagorean-theorem for real value functions using the L2 norm:

$$\int_0^1 (f(t)+g(t))^2\,dt = \int_0^1 f(t)^2\,dt + \int_0^1 g(t)^2\,dt$$

$$\int_0^1 (f(t)^2 + 2f(t)g(t) + g(t)^2)\,dt = \int_0^1 f(t)^2\,dt + \int_0^1 g(t)^2\,dt$$

$$\int_0^1 f(t)^2\,dt + \underbrace{2\int_0^1 f(t)g(t)\,dt}_{=0} + \int_0^1 g(t)^2\,dt = \int_0^1 f(t)^2\,dt + \int_0^1 g(t)^2\,dt$$

The orthogonality condition in L2[0,1] has to be $\int_0^1 f(t)g(t)\,dt = 0$

We define the inner product to be: $(f,g) = \int_0^1 f(t)g(t)\,dt \quad (f,f) = \int_0^1 f(t)^2\,dt = \|f\|^2$

Chose inner product by preserving validity of Pithagoream-theorem in the complex valued function space

First let’s restrict the function space to the Lebesgue functions satisfying:

$$\int_0^1 |f(t)|^2 \, dt < \infty$$

Requirements for an inner product in the complex domain

1. $(f, g) = \overline{(g, f)}$ (*Hermitian* symmetry)
2. $(f, f) \geq 0$ and $(f, f) = 0$ if and only if $f = 0$ (positive definiteness — same as before)
3. $(\alpha f, g) = \alpha(f, g)$, $(f, \alpha g) = \overline{\alpha}(f, g)$ (homogeneity — same as before in the first slot, conjugate scalar comes out if it’s in the second slot)
4. $(f + g, h) = (f, h) + (g, h)$, $(f, g + h) = (f, g) + (f, h)$ (additivity — same as before, no difference between additivity in first or second slot)

Chose inner product by preserving validity of Pithagoream-theorem in the complex valued function space

The Pithagorean-theorem:

$$\int_0^1 |f(t)+g(t)|^2 = \int_0^1 |f(t)|^2\,dt + \int_0^1 |g(t)|^2\,dt$$

$$\int_0^1 (|f(t)|^2 + 2\,\mathrm{Re}\{f(t)\overline{g(t)}\} + |g(t)|^2)\,dt = \int_0^1 |f(t)|^2\,dt + \int_0^1 |g(t)|^2\,dt$$

$$\int_0^1 |f(t)|^2\,dt + \underbrace{2\,\mathrm{Re}\left(\int_0^1 f(t)\overline{g(t)}\,dt\right)}_{=0} + \int_0^1 |g(t)|^2\,dt = \int_0^1 |f(t)|^2\,dt + \int_0^1 |g(t)|^2\,dt$$

The orthogonality condition in L2[0,1] can be $\int_0^1 f(t)\overline{g(t)}\,dt = 0$

We define the inner product to be: $(f,g) = \int_0^1 f(t)\overline{g(t)}\,dt$

$$(f,f) = \int_0^1 f(t)\overline{f(t)}\,dt = \int_0^1 |f(t)|^2\,dt = \|f\|^2$$

If $\int_0^1 |f(t)|^2\,dt < \infty$ and $\int_0^1 |g(t)|^2\,dt < \infty$ holds, then $(f,g) < \infty$

Determination of the coefficients of the Fourier-series of a function f(t) with period 1

We would like to write the function $f(t)$ as linear combination of exponentials with different frequencies:

$$f(t) \approx \sum_{m=-N}^{N} c_m e^{i2\pi mt}$$

Get the coefficients of the Fourier-series of $f(t)$ with the projection theory:

$$\sum_{n=-N}^{N} c_n \underbrace{\langle e_n, e_m \rangle}_{G_{nm}} = \underbrace{\langle f, e_m \rangle}_{b_m}$$

Basis functions:

$$e_m(t) = e^{i2\pi mt} \qquad m = 0, \pm 1, \pm 2..$$

Coefficients: c_n =?

As learned, by defining in such a way the coefficients (derived from orthogonality of the error to the approximating subspace) we minimise the norm of the error:

$$\left\| \sum_{m=-N}^{N} c_m e^{i2\pi mt} - f(t) \right\|$$

Please remember,
we defined the inner product to be: $\langle f, g \rangle = \int_0^1 f(t)\overline{\overline{g(t)}}dt$

And the induced norm to be: $\|g\| = \sqrt{\langle g, g \rangle} = \sqrt{\int_0^1 g(t)\overline{\overline{g(t)}}dt}$

Determination of the coefficients of the Fourier-series of a function f(t) with period 1

By projection theory, we project the function f(t) to the approximating subspace spanned by the basis functions:

$n \neq m$

$$G_{nm} = (e_n, e_m) = \int_0^1 e^{2\pi i n t}\overline{e^{2\pi i m t}}\, dt = \int_0^1 e^{2\pi i n t} e^{-2\pi i m t}\, dt = \int_0^1 e^{2\pi i (n-m) t}\, dt$$

$$= \frac{1}{2\pi i(n-m)} e^{2\pi i(n-m)t}\Big]_0^1 = \frac{1}{2\pi i(n-m)}\left(e^{2\pi i(n-m)} - e^0\right) = \frac{1}{2\pi i(n-m)}(1-1) = 0$$

$n = m$

$$G_{nm} = (e_n, e_n) = \int_0^1 e^{2\pi i n t}\overline{e^{2\pi i n t}}\, dt = \int_0^1 e^{2\pi i n t} e^{-2\pi i n t}\, dt = \int_0^1 e^{2\pi i (n-n) t}\, dt = \int_0^1 1\, dt = 1$$

$$\Rightarrow \quad G_{nm} = (e_n, e_m) = \delta_{nm} = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases} \quad \Rightarrow$$

The basis functions:

$e_m(t) = e^{i2\pi m t}$

are orthonormal

The gramian $\boldsymbol{G}$ is the identity matrix

Determination of the coefficients of the Fourier-series of a function f(t) with period 1

The gramian $\boldsymbol{G}$ is the identity matrix

$$\sum_{n=-N}^{N} c_n \underbrace{\langle e_n, e_m \rangle}_{G_{nm}} = \underbrace{\langle f, e_m \rangle}_{b_m} \quad \Rightarrow \quad \overbrace{\begin{bmatrix} 1 & & \\ & 1 & \\ & & \ddots \end{bmatrix}}^{\boldsymbol{G}} \overbrace{\begin{bmatrix} c_{-N} \\ c_{-N-1} \\ \vdots \end{bmatrix}}^{\boldsymbol{c}} = \overbrace{\begin{bmatrix} \langle f, e_{-N} \rangle \\ \langle f, e_{-N} \rangle \\ \vdots \end{bmatrix}}^{\boldsymbol{b}}$$

Coefficients:

$$c_m = \langle f, e_m \rangle = \int_0^1 f(t)\overline{e^{i2\pi mt}}dt = \int_0^1 f(t)e^{-i2\pi mt}dt$$

Please note, common notation for the Fourier-coefficient c_n of $f(t)$ is $\hat{f}(n)$:

$$f(t) \approx \sum_{m=-N}^{N} \hat{f}(n)e^{i2\pi mt} \qquad \hat{f}(n) = \int_0^1 f(t)e^{-i2\pi mt}dt$$

Connection between expension with sin and cos functions and the more general Fourier-series

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(2\pi n t) + \sum_{n=1}^{\infty} b_n \sin(2\pi n t) \quad \overset{?}{\Longleftrightarrow} \quad f(t) = \sum_{n=-\infty}^{\infty} c_n e^{i2\pi n t}$$

Using the identities:

$$\cos t = \frac{e^{it} + e^{-it}}{2}, \quad \sin t = \frac{e^{it} - e^{-it}}{2i}.$$

$$\cos(2\pi n t) = \frac{e^{2\pi i n t} + e^{-2\pi i n t}}{2}, \quad \sin(2\pi n t) = \frac{e^{2\pi i n t} - e^{-2\pi i n t}}{2i}$$

$$f(t) = \frac{a_0}{2}\left(\frac{e^{i2\pi 0 t} + e^{-i2\pi 0 t}}{2}\right) + \sum_{n=1}^{\infty} a_n \left(\frac{e^{i2\pi n t} + e^{-i2\pi n t}}{2}\right) + \sum_{n=1}^{\infty} b_n \left(\frac{e^{i2\pi n t} - e^{-i2\pi n t}}{2i}\right)$$

$$f(t) = \frac{a_0}{2} e^{i2\pi 0 t} + \sum_{n=1}^{\infty} \left(\frac{a_n}{2} + \frac{b_n}{2i}\right) e^{i2\pi n t} + \sum_{n=1}^{\infty} \left(\frac{a_n}{2} - \frac{b_n}{2i}\right) e^{-i2\pi n t}$$

Connection between expension with sin and cos functions and the more general Fourier-series

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos(2\pi nt)+\sum_{n=1}^{\infty}b_n\sin(2\pi nt) \quad \overset{?}{\Longleftrightarrow} \quad f(t)=\sum_{n=-\infty}^{\infty}c_n e^{i2\pi nt}$$

$$f(t)=\frac{a_0}{2}e^{i2\pi 0t}+\sum_{n=1}^{\infty}\left(\frac{a_n}{2}+\frac{b_n}{2i}\right)e^{i2\pi nt}+\sum_{n=1}^{\infty}\left(\frac{a_n}{2}-\frac{b_n}{2i}\right)e^{-i2\pi nt}$$

$$f(t)=\underbrace{\frac{a_0}{2}}_{c_0}e^{i2\pi 0t}+\sum_{n=1}^{\infty}\underbrace{\left(\frac{a_n}{2}+\frac{b_n}{2i}\right)}_{c_n}e^{i2\pi nt}+\sum_{n=-1}^{-\infty}\underbrace{\left(\frac{a_n}{2}-\frac{b_n}{2i}\right)}_{c_{-n}}e^{i2\pi nt}$$

$$f(t)=\sum_{n=-\infty}^{\infty}c_n e^{i2\pi nt}$$

$$c_n=\begin{cases}\frac{a_n}{2} & n=0\\ \left(\frac{a_n}{2}+\frac{b_n}{2i}\right) & n=1,2,..\\ \left(\frac{a_n}{2}-\frac{b_n}{2i}\right) & n=-1,-2,..\end{cases}$$

Properties of the Fourier-series: Fourier-series of real functions

$$c_n = \begin{cases} \frac{a_n}{2} & n = 0 \\ \left(\frac{a_n}{2} + \frac{b_n}{2i}\right) & n = 1,2,.. \\ \left(\frac{a_n}{2} - \frac{b_n}{2i}\right) & n = -1,-2,.. \end{cases}$$

If the function $f(t)$ is real: a_n and b_n have to be real.

$$Real(c_n) = \frac{a_n}{2} \qquad Im(c_n) = \begin{cases} 0 & n = 0 \\ \left(\frac{b_n}{2i}\right) & n = 1,2,.. \\ \left(-\frac{b_n}{2i}\right) & n = -1,-2,.. \end{cases}$$

The coefficients are conjugates: $c_{-n} = \overline{c_n}$

If the coefficients are conjugates: $c_{-n} = \overline{c_n}$

$$f(t) = \sum_{n=-N}^{N} c_n e^{i2\pi nt} = c_0 e^{i2\pi 0t} + \sum_{n=1}^{N} c_n e^{i2\pi nt} + c_{-n} e^{-i2\pi nt}$$

$$\overline{f(t)} = c_0 e^{i2\pi 0t} + \sum_{n=1}^{N} c_n e^{i2\pi nt} + c_{-n} e^{-i2\pi nt} = c_0 e^{i2\pi 0t} + \sum_{n=1}^{N} \overline{c_n e^{i2\pi nt}} + \overline{c_{-n} e^{-i2\pi nt}}$$

$$\overline{f(t)} = c_0 e^{i2\pi 0t} + \sum_{n=1}^{N} \overline{c_n}\,\overline{e^{i2\pi nt}} + \overline{c_{-n}}\,\overline{e^{-i2\pi nt}} = c_0 e^{i2\pi 0t} + \sum_{n=1}^{N} c_{-n} e^{-i2\pi nt} + c_n e^{i2\pi nt} = f(t)$$

The expension is real:

$$\overline{f(t)} = f(t)$$

Properties of the Fourier-series: Fourier-series of even functions

If the function $f(t)$ is even and real: $f(t) = f(-t)$

$$\overline{\hat{f}(n)} = \hat{f}(-n) = \int_0^1 e^{-2\pi i(-n)t} f(t)\, dt = \int_0^1 e^{2\pi int} f(t)\, dt$$

$$= -\int_0^{-1} e^{-2\pi ins} f(-s)\, ds \quad \text{(substituting } t = -s \text{ and changing limits accordingly)}$$

$$= \int_{-1}^{0} e^{-2\pi ins} f(s)\, ds \quad \text{(flipping the limits and using that } f(t) \text{ is even)}$$

$$= \hat{f}(n) \quad \text{(because you can integrate over any period, in this case from } -1 \text{ to } 0\text{)}$$

$$\hat{f}(n) = \overline{\hat{f}(n)} = \hat{f}(-n)$$

The coefficients are real

The same comes out with sin and cos functions:
If the function $f(t)$ is even, the sin terms have to cancel → $b_n = 0$

$$c_n = \begin{cases} \frac{a_n}{2} \\ \left(\frac{a_n}{2} + \frac{b_n}{2i}\right) = \frac{a_n}{2} = Real(c_n) \\ \left(\frac{a_n}{2} - \frac{b_n}{2i}\right) \end{cases}$$

Properties of the Fourier-series: existence of the expansion, calculation of energy from coeffs

energy $\int_0^1 |f(t)|^2 \, dt$

If the function is $L2[0,1]$ (in the Lebesgue-space): $\int_0^1 |f(t)|^2 \, dt < \infty$

then the integral defining the coefficients of the Fourier series: $\int_0^1 e^{-2\pi int} f(t) \, dt$

exists, and then

$$\lim_{N\to\infty} \left\| \sum_{n=-N}^{N} \hat{f}(n) e^{2\pi int} - f(t) \right\| = 0$$

Furthermore, if the basis is orthonormal, the above defined energy term can be simply calculated from the coefficients:

$$\int_0^1 |f(t)|^2 \, dt = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2$$

So the magnitude $|\hat{f}(n)|^2$ is the energy contributed by the n-th harmonic:

Fourier-series of a function f(t) with period different from 1

1.6.1 What if the period isn't 1?

Changing to a base period other than 1 does not present too stiff a challenge, and it brings up a very important phenomenon. If we're working with functions $f(t)$ with period T, then

$$g(t) = f(Tt)$$

has period 1. Suppose we have

$$g(t) = \sum_{n=-N}^{N} c_n e^{2\pi i n t},$$

or even, without yet addressing issues of convergence, an infinite series

$$g(t) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n t}.$$

Write $s = Tt$, so that $g(t) = f(s)$. Then

$$f(s) = g(t) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n t} = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n s/T}$$

Fourier-series of a function f(t) with period different from 1

The harmonics are now $e^{2\pi ins/T}$.

What about the coefficients? If

$$\hat{g}(n) = \int_0^1 e^{-2\pi int} g(t)\, dt$$

then, making the same change of variable $s = Tt$, the integral becomes

$$\frac{1}{T} \int_0^T e^{-2\pi ins/T} f(s)\, ds\,.$$

To wrap up, calling the variable t again, the Fourier series for a function $f(t)$ of period T is

$$\sum_{n=-\infty}^{\infty} c_n e^{2\pi int/T}$$

where the coefficients are given by

$$c_n = \frac{1}{T} \int_0^T e^{-2\pi int/T} f(t)\, dt\,.$$

Fourier-series of a function f(t) with period different from 1
Orthogonality of the new basis functions

$$(f,g)=\int_0^T f(t)\overline{g(t)}\,dt$$

$$\begin{aligned}(e^{2\pi int/T},e^{2\pi imt/T}) &= \int_0^T e^{2\pi int/T}\,\overline{e^{2\pi imt/T}}\,dt = \int_0^T e^{2\pi int/T}\,e^{-2\pi imt/T}\,dt \\ &= \int_0^T e^{2\pi i(n-m)t/T}\,dt = \frac{1}{2\pi i(n-m)/T}\,e^{2\pi i(n-m)t/T}\Big]_0^T \\ &= \frac{1}{2\pi i(n-m)/T}\,(e^{2\pi i(n-m)}-e^0) = \frac{1}{2\pi i(n-m)/T}\,(1-1)=0\end{aligned}$$

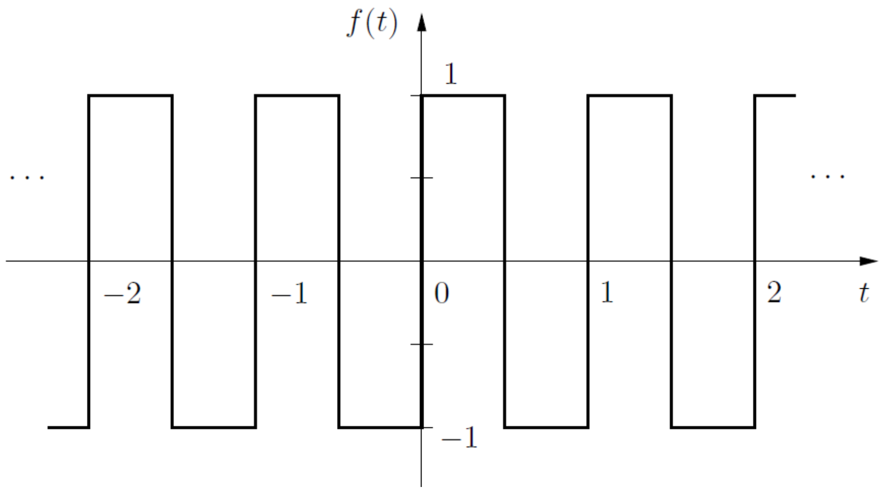
And when $n=m$:

$$\begin{aligned}(e^{2\pi int/T},e^{2\pi int/T}) &= \int_0^T e^{2\pi int/T}\,\overline{e^{2\pi int/T}}\,dt \\ &= \int_0^T e^{2\pi int/T}\,e^{-2\pi int/T}\,dt = \int_0^T 1\,dt = T\,.\end{aligned}$$

The basis functions: $e_m(t)=e^{i2\pi mt/T}$ are orthogonal, but not orthonormal

Examples
Fourier expansion of function $f(t)$ that is not in C_0

Consider a *square wave* of period 1, such as illustrated below.



$$f(t) = \sum_{n=-N}^{N} c_n e^{2\pi i n t} \qquad c_n = \int_0^1 e^{-2\pi i n t} f(t)\, dt \qquad \hat{f}(n) = \int_0^1 e^{-2\pi i n t} f(t)\, dt$$

$$f(t) = \begin{cases} +1 & 0 \le t < \frac{1}{2} \\ -1 & \frac{1}{2} \le t < 1 \end{cases}$$

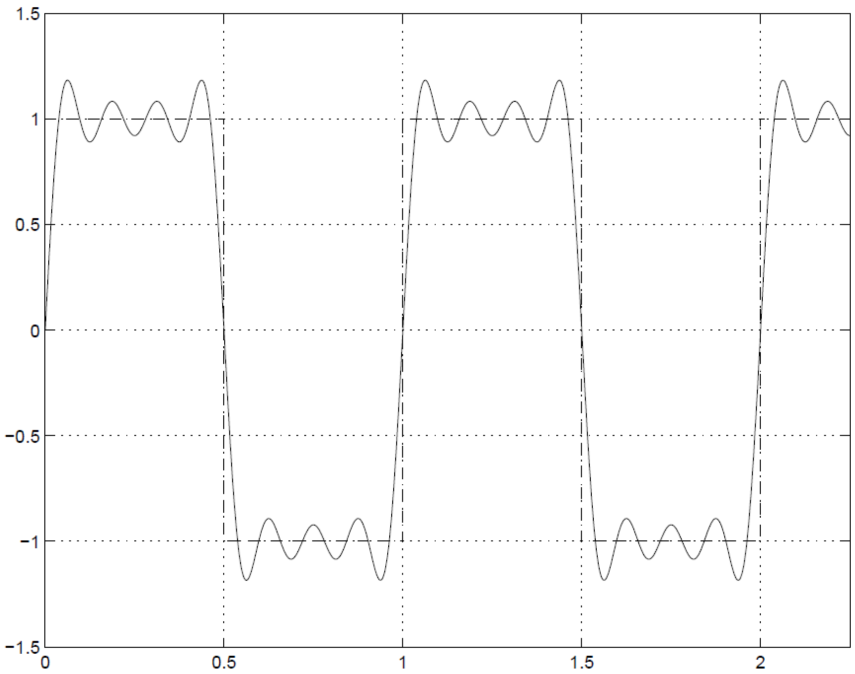
$$f(t) = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{1}{2k+1} \sin 2\pi(2k+1)t$$

$$\begin{aligned} \hat{f}(n) &= \int_0^1 e^{-2\pi i n t} f(t)\, dt \\ &= \int_0^{1/2} e^{-2\pi i n t}\, dt - \int_{1/2}^{1} e^{-2\pi i n t}\, dt \\ &= \left[-\frac{1}{2\pi i n} e^{-2\pi i n t}\right]_0^{1/2} - \left[-\frac{1}{2\pi i n} e^{-2\pi i n t}\right]_{1/2}^{1} = \frac{1}{\pi i n}\left(1 - e^{-\pi i n}\right) \end{aligned}$$

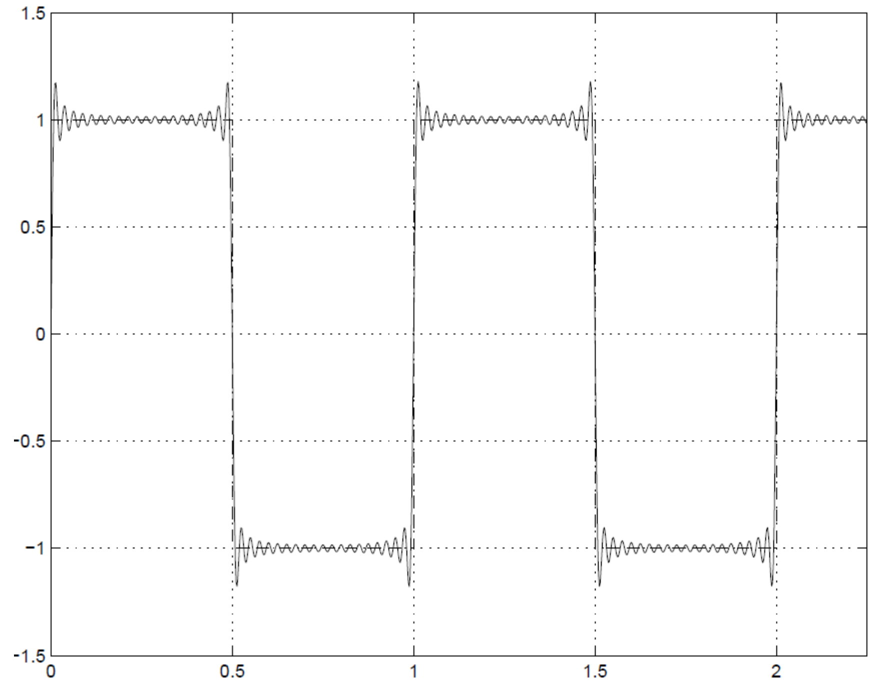
$$1 - e^{-\pi i n} = \begin{cases} 0 & n \text{ even} \\ 2 & n \text{ odd} \end{cases}$$

Examples
Fourier expansion of function $f(t)$ that is not in C_0

$$f(t) = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{1}{2k+1} \sin 2\pi(2k+1)t$$



$n = 1..9$



$n = 1..39$

Examples
Fourier expansion of function $f(t)$ that is not in C_1

$$f(t) = \begin{cases} \frac{1}{2} + t & -\frac{1}{2} \le t \le 0 \\ \frac{1}{2} - t & 0 \le t \le +\frac{1}{2} \end{cases}$$

$$\frac{1}{4} + \sum_{k=0}^{\infty} \frac{2}{\pi^2(2k+1)^2} \cos(2\pi(2k+1)t)$$