Technische Universität Braunschweig

Lecture 4:

Fourier series
Analytical solution of ODEs and PDEs,
The Finite Difference Method

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Overview of the course

- Introduction (definition of PDEs, classification, basic math, introductory examples of PDEs)
- Analytical solution of elementary PDEs, uniqueness and existence of the solution
- Numerical solutions of PDEs:
  - Finite difference method
  - Finite element method

Overview of this lecture

- I. Fourier series in the complex domain, further notes on projection theory
- II. Solving PDEs, analytical solution of ODEs
  - About existence and uniqueness of linear PDEs
  - Solution methods
    - Spectral method (Fourier analysis)
  - Essesntial ODEs
    - Solving homogenous second order ODEs
    - From homogenous to inhomogenous equation
    - Converting higher order ODEs to system of first order ODEs
    - Solving system of ODEs

Some more information on

norms, inner products and projection theory

Chose inner product by preserving validity of Pithagoream-theorem in the real valued function space

The Pithagorean-theorem: $\|f+g\|^2 = \|f\|^2 + \|g\|^2$

The Pithagorean-theorem for real value functions using the L2 norm:

$$\int_0^1 (f(t)+g(t))^2\,dt = \int_0^1 f(t)^2\,dt + \int_0^1 g(t)^2\,dt$$

$$\int_0^1 (f(t)^2 + 2f(t)g(t) + g(t)^2)\,dt = \int_0^1 f(t)^2\,dt + \int_0^1 g(t)^2\,dt$$

$$\int_0^1 f(t)^2\,dt + \underbrace{2\int_0^1 f(t)g(t)\,dt}_{=0} + \int_0^1 g(t)^2\,dt = \int_0^1 f(t)^2\,dt + \int_0^1 g(t)^2\,dt$$

The orthogonality condition in L2[0,1] has to be $\int_0^1 f(t)g(t)\,dt = 0$

We define the inner product to be: $(f,g) = \int_0^1 f(t)g(t)\,dt \quad (f,f) = \int_0^1 f(t)^2\,dt = \|f\|^2$

Chose inner product by preserving validity of Pithagoream-theorem in the complex valued function space

First let's restrict the function space to the Lebesgue functions satisfying:

$$\int_0^1 |f(t)|^2 \, dt < \infty$$

Requirements for an inner product in the complex domain

1. $(f, g) = \overline{(g, f)}$ (*Hermitian* symmetry)
2. $(f, f) \geq 0$ and $(f, f) = 0$ if and only if $f = 0$ (positive definiteness — same as before)
3. $(\alpha f, g) = \alpha(f, g)$, $(f, \alpha g) = \overline{\alpha}(f, g)$ (homogeneity — same as before in the first slot, conjugate scalar comes out if it's in the second slot)
4. $(f + g, h) = (f, h) + (g, h)$, $(f, g + h) = (f, g) + (f, h)$ (additivity — same as before, no difference between additivity in first or second slot)

Chose inner product by preserving validity of Pithagoream-theorem in the complex valued function space

The Pithagorean-theorem:

$$\int_0^1 |f(t)+g(t)|^2 = \int_0^1 |f(t)|^2\,dt + \int_0^1 |g(t)|^2\,dt$$

$$\int_0^1 (|f(t)|^2 + 2\,\mathrm{Re}\{f(t)\overline{g(t)}\} + |g(t)|^2)\,dt = \int_0^1 |f(t)|^2\,dt + \int_0^1 |g(t)|^2\,dt$$

$$\int_0^1 |f(t)|^2\,dt + \underbrace{2\,\mathrm{Re}\left(\int_0^1 f(t)\overline{g(t)}\,dt\right)}_{=0} + \int_0^1 |g(t)|^2\,dt = \int_0^1 |f(t)|^2\,dt + \int_0^1 |g(t)|^2\,dt$$

The orthogonality condition in L2[0,1] can be $\int_0^1 f(t)\overline{g(t)}\,dt = 0$

We define the inner product to be: $(f,g) = \int_0^1 f(t)\overline{g(t)}\,dt$

$$(f,f) = \int_0^1 f(t)\overline{f(t)}\,dt = \int_0^1 |f(t)|^2\,dt = \|f\|^2$$

If $\int_0^1 |f(t)|^2\,dt < \infty$ and $\int_0^1 |g(t)|^2\,dt < \infty$ holds, then $(f,g) < \infty$

Determination of the coefficients of the Fourier-series of a function f(t) with period 1

We would like to write the function $f(t)$ as linear combination of exponentials with different frequencies:

$$f(t) \approx \sum_{m=-N}^{N} c_m e^{i2\pi mt}$$

Get the coefficients of the Fourier-series of $f(t)$ with the projection theory:

$$\sum_{n=-N}^{N} c_n \underbrace{\langle e_n, e_m \rangle}_{G_{nm}} = \underbrace{\langle f, e_m \rangle}_{b_m}$$

Basis functions:

$$e_m(t) = e^{i2\pi mt} \qquad m = 0, \pm 1, \pm 2..$$

Coefficients: c_n =?

As learned, by defining in such a way the coefficients (derived from orthogonality of the error to the approximating subspace) we minimise the norm of the error:

$$\left\| \sum_{m=-N}^{N} c_m e^{i2\pi mt} - f(t) \right\|$$

Please remember,
we defined the inner product to be: $\langle f, g \rangle = \int_0^1 f(t)\overline{\overline{g(t)}}dt$

And the induced norm to be: $\|g\| = \sqrt{\langle g, g \rangle} = \sqrt{\int_0^1 g(t)\overline{\overline{g(t)}}dt}$

Determination of the coefficients of the Fourier-series of a function f(t) with period 1

By projection theory, we project the function f(t) to the approximating subspace spanned by the basis functions:

$n \neq m$

$$G_{nm} = (e_n, e_m) = \int_0^1 e^{2\pi int}\overline{e^{2\pi imt}}\,dt = \int_0^1 e^{2\pi int}e^{-2\pi imt}\,dt = \int_0^1 e^{2\pi i(n-m)t}\,dt$$

$$= \frac{1}{2\pi i(n-m)}e^{2\pi i(n-m)t}\Big]_0^1 = \frac{1}{2\pi i(n-m)}\left(e^{2\pi i(n-m)} - e^0\right) = \frac{1}{2\pi i(n-m)}(1-1) = 0$$

$n = m$

$$G_{nm} = (e_n, e_n) = \int_0^1 e^{2\pi int}\overline{e^{2\pi int}}\,dt = \int_0^1 e^{2\pi int}e^{-2\pi int}\,dt = \int_0^1 e^{2\pi i(n-n)t}\,dt = \int_0^1 1\,dt = 1$$

$$\Rightarrow \quad G_{nm} = (e_n, e_m) = \delta_{nm} = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases} \quad \Rightarrow$$

The basis functions:

$e_m(t) = e^{i2\pi mt}$

are orthonormal

The gramian $\boldsymbol{G}$ is the identity matrix

Determination of the coefficients of the Fourier-series of a function f(t) with period 1

The gramian $\boldsymbol{G}$ is the identity matrix

$$\sum_{n=-N}^{N} c_n \underbrace{\langle e_n, e_m \rangle}_{G_{nm}} = \underbrace{\langle f, e_m \rangle}_{b_m} \Rightarrow \underbrace{\begin{bmatrix} 1 & & \\ & 1 & \\ & & \ddots \end{bmatrix}}_{\boldsymbol{G}} \underbrace{\begin{bmatrix} c_{-N} \\ c_{-N-1} \\ \vdots \end{bmatrix}}_{\boldsymbol{c}} = \underbrace{\begin{bmatrix} \langle f, e_{-N} \rangle \\ \langle f, e_{-N} \rangle \\ \vdots \end{bmatrix}}_{\boldsymbol{b}}$$

Coefficients:

$$c_m = \langle f, e_m \rangle = \int_0^1 f(t)\overline{e^{i2\pi mt}}\,dt = \int_0^1 f(t)e^{-i2\pi mt}\,dt$$

Please note, common notation for the Fourier-coefficient c_n of $f(t)$ is $\hat{f}(n)$:

$$f(t) \approx \sum_{m=-N}^{N} \hat{f}(n)e^{i2\pi mt} \qquad \hat{f}(n) = \int_0^1 f(t)e^{-i2\pi mt}\,dt$$

II.

Existence, uniqueness, esssential of ODEs, solution methods, the spectral method

Essential ODEs
solving linear systems↔analytical solution of linear ODEs

$$\mathbf{Ax} = \mathbf{b}$$

Existence:

$\mathbf{b} \in R(\mathbf{C})$ ($\mathbf{b}$ is in the range of $\mathbf{A}$)

Uniqueness:
let's suppose $\mathbf{y}$ and $\boldsymbol{z}$ are both solutions:

$$\mathbf{Ay} = \mathbf{b} \qquad \mathbf{Az} = \mathbf{b}$$

$\mathbf{A}(\mathbf{y} - \mathbf{z}) = \mathbf{0}$ ➡ if $\mathbf{y} \neq \boldsymbol{z}$ nontrivial solution

In other words, the nullspace of $\mathbf{A}$ is nontrivial.

The system has only unique solution if the nullspace of $\mathbf{A}$ is trivial, that is the only solution of
$\mathbf{Ax} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$

$$Lu = f$$

example: $L_D u = \dfrac{\partial^2 u}{\partial x^2}$

Existence:

$u \in R(L)$ (f is in the range of L)

Uniqueness:
let's suppose y and z are both solutions:

$$Ly = f \qquad Lz = f$$

$L(y - z) = 0$ ➡ if $y \neq z$ nontrivial solution

The system has only unique solution if the nullspace of L is trivial, that is the only solution of
$Lu = 0$ is the zero function

Essential ODEs
solving linear systems↔analytical solution of linear ODEs

$$\mathbf{Ax} = \mathbf{b}$$

<u>Solution:</u>

If $N(\mathbf{A})$ is nontrivial, it has only solution if it satisfies a certain compatibility solution:
Adjoint operator: $\mathbf{A}^{\mathrm{T}} \rightarrow \langle \mathbf{Ax}, \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{A}^{\mathrm{T}}\mathbf{y} \rangle$

$$\mathbf{A}^{\mathrm{T}}\mathbf{w} = \mathbf{0} \quad \Rightarrow \quad \mathbf{w} \in N(\mathbf{A}^{\mathrm{T}})$$

$$\mathbf{w} \cdot \mathbf{b} = \mathbf{0}$$

If $N(\mathbf{A})$ is nontrivial, and if it has a solution, it has infinitely many:

$$\left.\begin{aligned} \mathbf{Aw} &= \mathbf{0} \\ \mathbf{Az} &= \mathbf{b} \end{aligned}\right\} \quad \mathbf{A}(\mathbf{z} + \alpha\mathbf{w}) = \mathbf{Az} + \alpha\mathbf{Aw} = \mathbf{b}$$

⇒ $\mathbf{z} + \alpha\mathbf{w}$ is also a solution

$$Lu = f$$

<u>Solution:</u>

If $N(L)$ is nontrivial, it has only solution if it satisfies a certain compatibility solution.

Adjoint operator L^*: $\quad \langle Lu, v \rangle = \langle u, L^* v \rangle$

If $N(L)$ is nontrivial, and if it has a solution, it has infinitely many:

$$\left.\begin{aligned} Lw &= 0 \\ Lz &= f \end{aligned}\right\} \quad L(z + \alpha w) = Lz + \alpha Lw = f$$

⇒ $z + \alpha w$ is also a solution

Essential ODEs
solving linear systems↔analytical solution of linear ODEs

Uniqueness (example1):

$$L_D u = -\alpha \frac{\partial^2 u}{\partial x^2} \Rightarrow \left.\begin{array}{l} -\alpha \frac{\partial^2 u(x)}{\partial x^2} = f(x)\ x \in [0,l] \\ u(0) = 0 \\ u(l) = 0 \end{array}\right\} L_D : C_D^2[0,l] \to C[0,l]$$

The homogenous system: $-\alpha \frac{\partial^2 u(x)}{\partial x^2} = 0$

$\Rightarrow u(x) = ax + b$ $\quad \left.\begin{array}{l} u(0) = 0 \Rightarrow b = 0 \\ u(l) = 0 \Rightarrow a = 0 \end{array}\right\} \Rightarrow u(x) = 0 \Rightarrow$ the trivial solution unique solution of $L_D u = f$

$$-\alpha \frac{d^2 u(x)}{dx^2} = f(x) \Rightarrow \alpha \frac{du(x)}{dx} = -\overbrace{\int_0^x f(s) ds}^{F(x)} + c_1 \Rightarrow \alpha u = -\int_0^x F(s) ds + c_1 x + c_2$$

$$u(0) = 0 \Rightarrow c_2 = 0$$

$$u(l) = 0 \Rightarrow c_1 = \frac{1}{l} \int_0^l \int_0^z f(s) ds dz$$

Essential ODEs
solving linear systems↔analytical solution of linear ODEs

Uniqueness (example2) :

$$L_N u = f \Rightarrow \begin{cases} -\alpha u_{xx} = f(x) & x \in [0,l] \\ u_x(0) = 0 \\ u_x(l) = 0 \end{cases} \quad L_D: C_N^2[0,l] \to C[0,l]$$

The homogenous system:

$$-\alpha \frac{\partial^2 u(x)}{\partial x^2} = 0$$

$$\Rightarrow u(x) = ax + b \qquad \left.\begin{matrix} u_x(0) = 0 \Rightarrow a = 0 \\ u_x(l) = 0 \end{matrix}\right\} \Rightarrow u(x) = b \Rightarrow$$

non trivial solution
if there is a solution, it is not unique

$$-\alpha \frac{d^2 u(x)}{dx^2} = f(x) \Rightarrow -\alpha \left[\frac{du(x)}{dx}\right]_0^l = \int_0^l f(x)dx \Rightarrow \int_0^l f(x)dx = 0$$

Essential ODEs
solving linear systems↔analytical solution of linear ODEs

$\mathbf{Ax} = \mathbf{b}$

Solution:

1) **General solution**

$$\mathbf{A}^{-1}\mathbf{Ax} = \mathbf{A}^{-1}\mathbf{b} \qquad \mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

2) **Direct solvers (Gauß elimination), iterative methods**

3) **Spectral method**

If $\mathbf{A}^{\mathrm{T}} = \mathbf{A}$ (real eigenvalues) $\quad \mathbf{A}\mathbf{v}_{\mathrm{i}} = \lambda_i \mathbf{v}_{\mathrm{i}}$

$$\mathbf{b} = \sum_i (\mathbf{v}_{\mathbf{i}} \cdot \mathbf{b})\mathbf{v}_{\mathrm{i}} \qquad \mathbf{x} = \sum_i (\mathbf{v}_{\mathbf{i}} \cdot \mathbf{x})\mathbf{v}_{\mathrm{i}} = \sum_i \alpha_i \mathbf{v}_{\mathrm{i}}$$

$$\mathbf{Ax} = \mathbf{b} \Rightarrow \mathbf{A}\sum_i \alpha_i \mathbf{v}_{\mathrm{i}} = \sum_i (\mathbf{v}_{\mathbf{i}} \cdot \mathbf{b})\mathbf{v}_{\mathrm{i}}$$

$$\sum_i \alpha_i \underbrace{\mathbf{A}\mathbf{v}_{\mathrm{i}}}_{\lambda_i \mathbf{v}_{\mathrm{i}}} = \sum_i (\mathbf{v}_{\mathbf{i}} \cdot \mathbf{b})\mathbf{v}_{\mathrm{i}} \Rightarrow \alpha_i \lambda_i = (\mathbf{v}_{\mathbf{i}} \cdot \mathbf{b})$$

$$\mathbf{x} = \sum_i \frac{(\mathbf{v}_{\mathbf{i}} \cdot \mathbf{b})}{\lambda_i}\mathbf{v}_{\mathrm{i}}$$

$Lu = f$

Solution:

1) **Direct integration**
Method of Green's functions

2) **Galerkin method/FD method**

3) **Fourier series** $\quad Lv_i = \lambda v_i$

$\langle Lu, v\rangle = \langle u, Lv\rangle$ (real eigenvalues)

$$f = \sum_i f_i v_i(x)$$

$$u = \sum_i u_i v_i(x)$$

Essential ODEs
solving linear systems↔analytical solution of linear ODEs

Solving ODEs with **Fourier series - example** $-\alpha \frac{d^2u}{dx^2} = f(x)$ $u(0) = 0$, $u(l) = 0$ $L_D u = f(x)$

1) Solve the eigenvalues-eigenfunctions $(\lambda_i, v_i(x))$

a) Can eigenfunctions form an orthogonal basis (is the operator symmetric)? $\langle Lu, v\rangle = \langle u, Lv\rangle?$

$$\langle Lu, v\rangle = -\alpha \int_0^l \frac{d^2u(x)}{dx^2} v(x)dx = \left[-\alpha \frac{du(x)}{dx} v(x)\right]_0^l + \alpha \int_0^l \frac{du(x)}{dx}\frac{dv(x)}{dx} dx$$

$$= \alpha \int_0^l \frac{du(x)}{dx}\frac{dv(x)}{dx} dx = \left[\alpha u(x)\frac{dv(x)}{dx}\right]_0^l - \alpha \int_0^l u(x)\frac{dv(x)}{dx^2} dx =$$

$$= \alpha \int_0^l u(x)\frac{dv(x)}{dx^2} dx = \langle u, Lv\rangle$$

b) Find eigenfunctions and eigenvalues $L_D v_i = \lambda v_i(x)$

$v_i = \sin\left(\frac{i\pi x}{l}\right)$ ➡ We try to find the solution in the form $u = \sum_i u_i \sin\left(\frac{i\pi x}{l}\right)$

Essential ODEs
solving linear systems↔analytical solution of linear ODEs

Solving ODEs with **Fourier series - example** $-\alpha\frac{d^2u}{dx^2} = f(x)$ $\quad u(0) = 0$, $u(l) = 0$ $\quad L_D u = f(x)$

2) Project $f(x)$ to the space spanned by the eigenfunctions:

$$f(x) = \sum_i f_i \sin\left(\frac{i\pi x}{l}\right) \qquad f_i = \frac{\left\langle f_i, \sin\left(\frac{i\pi x}{l}\right)\right\rangle}{\left\langle \sin\left(\frac{i\pi x}{l}\right), \sin\left(\frac{i\pi x}{l}\right)\right\rangle}$$

3) Solve the ODE for u_i:

$$-\alpha\frac{d^2}{dx^2}\sum_i u_i \sin\left(\frac{i\pi x}{l}\right) = -\alpha\sum_i u_i \frac{d^2}{dx^2}\sin\left(\frac{i\pi x}{l}\right) = \sum_i \alpha\frac{i^2\pi^2}{l^2}u_i \sin\left(\frac{i\pi x}{l}\right) = \sum_i f_i \sin\left(\frac{i\pi x}{l}\right)$$

$$\alpha\frac{i^2\pi^2}{l^2}u_i = f_i \Rightarrow u_i = \frac{l^2 f_i}{i^2\pi^2\alpha} \qquad u(x) = \sum_i u_i \sin\left(\frac{i\pi x}{l}\right)$$