



Technische
Universität
Braunschweig



Introduction to PDEs and Numerical Methods

Lecture 3:

Analytical solution of ODEs and PDEs

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Overview of the course

- Introduction (definition of PDEs, classification, basic math, introductory examples of PDEs)
- Analytical solution of elementary PDEs (Fourier series/transform, separation of variables, Green's function)
- Numerical solutions of PDEs:
 - Finite difference method
 - Finite element method

Overview of this lecture

- Classification of PDEs revisited, some examples of PDEs again, boundary conditions ✓
- Solving linear systems – solving linear PDEs
- Eigenvalues, eigenfunctions, solving linear equations with spectral method – spectral method for PDEs
- Essential functional analysis, projection theory, Fourier series
- Separation of variables – analytical solution of the heat equation

Projection theory and the Fourier series

1.) Essential functional analysis - vector space

- **functional** is a function from a **vector space** into its underlying scalar field
- Vector spaces (V)

$$u, v \in V$$

- addition of u and v ($z = u + v$) is defined such that $z \in V$
- multiplication by scalars is defined such that $\alpha u \in V$

For addition and multiplication the following axioms are satisfied:

$$u + (v + w) = (u + v) + w \text{ (associativity)}$$

$$u + v = v + u \text{ (commutativity)}$$

There exists an element $\mathbf{0} \in V$, such that $v + \mathbf{0} = v$ for all $v \in V$.

For every $v \in V$, there exists an element $-v \in V$ such that $v + (-v) = \mathbf{0}$

$$a(bv) = (ab)v$$

$1v = v$, where 1 denotes the multiplicative identity in F .

$$a(u + v) = au + av \text{ (distributivity)}$$

$$(a + b)v = av + bv \text{ (distributivity)}$$

with:

$$u, v, w \in V$$

$$a, b \in \mathbb{R}$$

Projection theory and the Fourier series

1.) Essential functional analysis - vector spaces, linear subspaces

Important vector spaces, examples:

- Euclidian n-space:

$$\mathbf{R}^n = \{(u_1, u_2, \dots, u_n) : u_i \in \mathbf{R}, i = 1, 2, \dots, n\}$$

- $C[a, b]$: set of all continuous, real-valued functions defined on the interval $[a, b]$.
- $C^1[a, b]$: set of all real-valued, continuously differentiable functions defined on the interval $[a, b]$. (A function is continuously differentiable if its derivative exists and is continuous.)
- $C^k[a, b]$: space of real-valued functions defined on $[a, b]$ that have k continuous derivatives

Subspaces

linear subspaces $V_0 \in V, x, y \in V_0, a, b \in \mathbf{R}, ax + by \in V_0$

(any linear combination of two elements of the subspace is also an element of the subspace)

examples:

$$C_D^2[a, b] = \{u \in C^2[a, b] : u(a) = u(b) = 0\} . \quad \text{Dirichlet}$$

$$C_N^2[a, b] = \{u \in C^2[a, b] : \frac{du}{dx}(a) = \frac{du}{dx}(b) = 0\} \quad \text{Neumann}$$

Projection theory and the Fourier series

1.) Essential functional analysis - norms

A vector space may be endowed with additional structures, such as a **norm** and **inner product**

Norm in Euclidian spaces: any mapping $g: V \rightarrow \mathbb{R}$, that satisfies:

1. The zero vector, $\mathbf{0}$, has zero length; every other vector has a positive length.

$$g(\mathbf{x}) \geq 0, \quad g(\mathbf{x}) = 0 \text{ only if } \mathbf{x} = \mathbf{0}$$

with the usual notation

$$\|\mathbf{x}\| \geq 0, \quad \|\mathbf{x}\| = 0 \text{ only if } \mathbf{x} = \mathbf{0}$$

2. Multiplying a vector by a positive number changes its length without changing its direction.

$$g(a\mathbf{x}) = |a|g(\mathbf{x})$$

3. The triangle inequality holds. That is, taking norms as distances, the distance from point A through B to C is never shorter than going directly from A to C, or the shortest distance between any two points is a straight line.

$$g(\mathbf{x} + \mathbf{y}) \leq g(\mathbf{x}) + g(\mathbf{y})$$

Projection theory and the Fourier series

1.) Essential functional analysis - norms

Examples for norms in the Eucliden space

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\|\mathbf{x}\|_1 = |x_1| + |x_2| + |x_3|$$

$$\|\mathbf{x}\|_2 = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

$$\|\mathbf{x}\|_p = (|x_1|^p + |x_2|^p + |x_3|^p)^{\frac{1}{p}}$$

$$\|\mathbf{x}\|_\infty = \max(|x_1|, |x_2|, |x_3|)$$

Projection theory and the Fourier series

1.) Essential functional analysis - norms

Norm in function spaces, equivalently:

any mapping $g: V \rightarrow \mathbb{R}$, that satisfies:

1.

$$g(f(x)) \geq 0, \quad g(f(x)) = 0 \text{ only if } f(x) = 0$$

with the usual notation

$$\|f(x)\| \geq 0, \quad \|f(x)\| = 0 \text{ only if } f(x) = 0$$

2.

$$g(af(x)) = |a|g(f(x))$$

3.

$$g(f(x) + h(x)) \leq g(f(x)) + g(h(x))$$

Examples:

$$\|f\|_2 = \sqrt{\int_0^1 f(x)^2 dx} \quad \|f\|_p = \left(\int_0^1 |f(x)|^p dx \right)^{\frac{1}{p}}$$
$$\|f\|_\infty = \max_{x \in [0,1]} (|f(x)|)$$

Projection theory and the Fourier series

1.) Essential functional analysis – inner products

Inner product space: vector space with an inner product

Inner product:

any mapping $g: V \times V \rightarrow \mathbb{R}$, that satisfies

1. Positivity

$$g(x, x) \geq 0, \quad g(x, x) = 0 \text{ only if } x = 0$$

with the usual notation

$$\langle x, x \rangle \geq 0, \quad \langle x, x \rangle = 0 \text{ only if } x = 0$$

or

$$(x, x) \geq 0, \quad (x, x) = 0 \text{ only if } x = 0$$

2. Linearity

$$g(\alpha x, y) = \alpha g(x, y)$$

$$g(x + y, z) = g(x, z) + g(y, z)$$

3. Symmetry

$$g(x, y) = g(y, x)$$

Projection theory and the Fourier series

1.) Essential functional analysis – inner products

Examples

In Euclidian space (\mathbb{R}^n)

- Dot product

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x} \cdot \mathbf{y} =$$

$$\langle (x_1, \dots, x_n), (y_1, \dots, y_n) \rangle := \mathbf{x}^T \mathbf{y} = \sum_{i=1}^n x_i y_i = x_1 y_1 + \dots + x_n y_n,$$

- Hermitian form

$$\langle \mathbf{x}, \mathbf{y} \rangle_A = \mathbf{x}^T \mathbf{A} \mathbf{y} \text{ if } \mathbf{A}: \text{ pos. def. symm.}$$

In function spaces

- $\langle f, g \rangle = \int_0^1 f(x)g(x)dx \quad C_1[0,1]$
- $\langle f, g \rangle_{H^1} = \int_0^1 f(x)g(x) + f'(x)g'(x)dx$

Projection theory and the Fourier series

2.) Projection theory

Approximation of a vector u

we try to find the coefficients α_j of an approximating vector in the form:

$$u_h = \sum_{j=1}^n \alpha_j v_j$$

where

v_j : known (linearly independent) vectors

u_h : the approximation of the vector u , which is in an n-dimensional space:

$$u_h \in V_h = \text{span}\{v_1, v_2, \dots, v_n\}$$

Approximation of a function $u(x)$

we try to find the coefficients α_j of a „proximal model“ (ansatz function):

$$u_h(x) = \sum_{j=1}^n \alpha_j \omega_j(x)$$

where

$\omega_j(x)$: known (linearly independent) basis or ansatz functions

$u_h(x)$: the approximation of the solution $u(x)$, which is in an n-dimensional space:

$$u_h \in V_h = \text{span}\{\omega_1, \omega_2, \dots, \omega_n\}$$

Projection theory and the Fourier series

2.) Projection theory

Approximation of a vector u

Approximation of a function $u(x)$

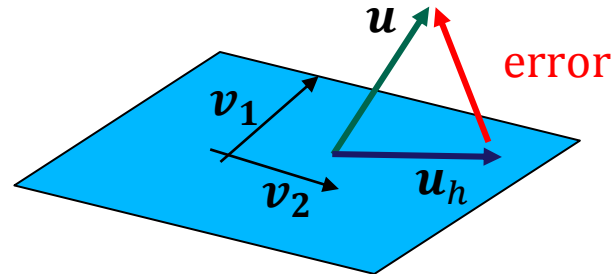
Our goal is to minimize the difference in between the solution and the approximation:

$$\text{error} = \|u - u_h\| < \|u - z\| \quad \forall z \in V_h$$

$$\text{error} = \|u(x) - u_h(x)\| < \|u(x) - z(x)\| \quad \forall z(x) \in V_h$$

The best approximation u_h to u from V_h is the one where the error is orthogonal to the space of V_h , that is to all possible $z \in V_h$.

Instead of writing it for all z (as z is an n -dimensional space) we can write



$$\forall v_j \quad j = 1..n$$

$$\langle (u - u_h), v_j \rangle = 0 \quad j = 1..n$$

$$\forall \omega_j \quad j = 1..n$$

$$\langle (u(x) - u_h(x)), \omega_j(x) \rangle = 0 \quad j = 1..n$$

Projection theory and the Fourier series

2.) Projection theory

Approximation of a vector u

Approximation of a function $u(x)$

Plugging in the proxi model to the orthogonality condition we have:

$$\left\langle \left(u - \sum_{i=1}^n \alpha_i v_i \right), v_j \right\rangle = 0 \quad j = 1..n \quad \Bigg| \quad \left\langle \left(u - \sum_{i=1}^n \alpha_i \omega_i \right), \omega_j \right\rangle = 0 \quad j = 1..n$$

Rearranging the equation we get:

$$\begin{array}{l} \langle u, v_j \rangle - \sum_{i=1}^n \alpha_i \langle v_i, v_j \rangle = 0 \quad j = 1..n \\ \sum_{i=1}^n \alpha_i \underbrace{\langle v_i, v_j \rangle}_{G_{ij}} = \underbrace{\langle u, v_j \rangle}_{b_j} \quad j = 1..n \end{array} \quad \Bigg| \quad \begin{array}{l} \langle u, \omega_j \rangle - \sum_{i=1}^n \alpha_i \langle \omega_i, \omega_j \rangle = 0 \quad j = 1..n \\ \sum_{i=1}^n \alpha_i \underbrace{\langle \omega_i, \omega_j \rangle}_{G_{ij}} = \underbrace{\langle u, \omega_j \rangle}_{b_j} \quad j = 1..n \end{array}$$

$$G\alpha = b$$

Projection theory and the Fourier series

3.) Fourier series

$f(y)$: some complicated function, we seek for a nicer interpretation in the domain $[-\pi, \pi]$ in the form:

$$f(x) = a_0 + a_1 \cos(x) + a_2 \cos(2x) + \dots \\ b_1 \sin(x) + b_2 \sin(2x) + \dots$$

- a_k, b_k coefficients? Let's minimise the error of the approximation using the inner product: $\langle f, g \rangle = \int_{-\pi}^{\pi} f(x) g(x) dx$ and the induced norm: $\sqrt{\langle f, f \rangle}$

First some important trigonometric identities:

$$\cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta \quad \Rightarrow \quad \cos(\alpha) \cos(\beta) = \frac{1}{2} [\cos(\alpha + \beta) + \cos(\alpha - \beta)]$$

$$\Rightarrow \quad \sin(\alpha) \sin(\beta) = \frac{1}{2} [\cos(\alpha - \beta) - \cos(\alpha + \beta)]$$

$$\Rightarrow \quad \cos^2(\alpha) = \frac{1}{2} [\cos(2\alpha) + 1]$$

$$\Rightarrow \quad \sin^2(\alpha) = \frac{1}{2} [1 - \cos(2\alpha)]$$

Projection theory and the Fourier series

3.) Fourier series

With the help of these identities, we can prove, that:

$$\int_{-\pi}^{\pi} \cos(kx) \sin(lx) dx = 0 \quad (1)$$

$$\int_{-\pi}^{\pi} \cos(kx) \cos(lx) dx = \begin{cases} 0 & \text{for } k \neq l \\ \pi & \text{for } k = l \end{cases} \quad (2)$$

$$\int_{-\pi}^{\pi} \sin(kx) \sin(lx) dx = \begin{cases} 0 & \text{for } k \neq l \\ \pi & \text{for } k = l \end{cases} \quad (3)$$

$$\cos(\alpha) \cos(\beta) = \frac{1}{2} [\cos(\alpha + \beta) + \cos(\alpha - \beta)]$$

$$\sin(\alpha) \sin(\beta) = \frac{1}{2} [\cos(\alpha - \beta) - \cos(\alpha + \beta)]$$

$$\cos^2(\alpha) = \frac{1}{2} [\cos(2\alpha) + 1]$$

$$\sin^2(\alpha) = \frac{1}{2} [1 - \cos(2\alpha)]$$

Projection theory and the Fourier series

3.) Fourier series

Get the **coefficients** of the Fourier series of $f(x)$ with the projection theory

$$\sum_{i=1}^n \alpha_i \underbrace{\langle \omega_i, \omega_j \rangle}_{G_{ij}} = \underbrace{\langle f, \omega_j \rangle}_{b_j} \quad j = 1..n$$

Coefficients: $\alpha_i = ?$

Basis functions:

$$\omega_k(x) = \begin{cases} \cos(kx) \\ \sin(kx) \\ 1 \end{cases} \quad k = 1, 2, \dots$$

	1	$\cos(x)$	$\sin(x)$	$\cos(2x)$	$\sin(2x)$...
1	$\int_{-\pi}^{\pi} 1 \cdot 1 dx$	$\int_{-\pi}^{\pi} 1 \cos(x) dx$	$\int_{-\pi}^{\pi} 1 \sin(x) dx$	$\int_{-\pi}^{\pi} 1 \cos(2x) dx$	$\int_{-\pi}^{\pi} 1 \sin(2x) dx$...
$\cos(x)$	$\int_{-\pi}^{\pi} \cos(x) dx$	$\int_{-\pi}^{\pi} \cos(x) \sin(x) dx$	$\int_{-\pi}^{\pi} \cos(x) \sin(x) dx$	$\int_{-\pi}^{\pi} \cos(x) \cos(2x) dx$	$\int_{-\pi}^{\pi} \cos(x) \sin(2x) dx$...
$\sin(x)$	$\int_{-\pi}^{\pi} \sin(x) dx$	$\int_{-\pi}^{\pi} \sin(x) \cos(x) dx$	\ddots			
\vdots		\vdots				

Projection theory and the Fourier series

3.) Fourier series

$$G = \begin{bmatrix} \int_{-\pi}^{\pi} 1 \cdot 1 dx & \int_{-\pi}^{\pi} 1 \cos(x) dx & \int_{-\pi}^{\pi} 1 \sin(x) dx & \int_{-\pi}^{\pi} 1 \cos(2x) dx & \int_{-\pi}^{\pi} 1 \sin(2x) dx & \dots \\ \int_{-\pi}^{\pi} \cos(x) dx & \int_{-\pi}^{\pi} \cos(x) \sin(x) dx & \int_{-\pi}^{\pi} \cos(x) \sin(x) dx & \int_{-\pi}^{\pi} \cos(x) \cos(2x) dx & \int_{-\pi}^{\pi} \cos(x) \sin(2x) dx & \dots \\ \int_{-\pi}^{\pi} \sin(x) dx & \int_{-\pi}^{\pi} \cos(x) \sin(x) dx & \ddots & & & \\ & \vdots & & & & \end{bmatrix}$$

$$\begin{aligned} \int_{-\pi}^{\pi} 1 \cdot 1 dx &= 2\pi & \int_{-\pi}^{\pi} \cos(kx) \sin(lx) dx &= 0 & \int_{-\pi}^{\pi} \cos(kx) \cos(lx) dx &= \begin{cases} 0 & \text{for } k \neq l \\ \pi & \text{for } k = l \end{cases} \\ \int_{-\pi}^{\pi} \sin(kx) \sin(lx) dx &= \begin{cases} 0 & \text{for } k \neq l \\ \pi & \text{for } k = l \end{cases} & \int_{-\pi}^{\pi} \sin(lx) dx &= 0 & \int_{-\pi}^{\pi} \cos(lx) dx &= \begin{cases} 0 & \text{for } l = 1, 2, 3, \dots \\ 2\pi & \text{for } l = 0 \end{cases} \end{aligned}$$

$$G = \begin{bmatrix} 2\pi & 0 & 0 & 0 & 0 & \dots \\ 0 & \pi & 0 & 0 & 0 & \dots \\ 0 & 0 & \pi & & & \\ \vdots & & & & & \end{bmatrix} \quad \mathbf{G}\boldsymbol{\alpha} = \mathbf{b} \quad \mathbf{b} = \langle f, \omega_j \rangle = \begin{bmatrix} \int_{-\pi}^{\pi} f \cdot 1 dx \\ \int_{-\pi}^{\pi} f \cos(x) dx \\ \int_{-\pi}^{\pi} f \sin(x) dx \\ \int_{-\pi}^{\pi} f \cos(2x) dx \\ \vdots \end{bmatrix}$$

Projection theory and the Fourier series

3.) Fourier series

$$\begin{bmatrix} 2\pi & 0 & 0 & 0 & 0 & \dots \\ 0 & \pi & 0 & 0 & 0 & \dots \\ 0 & 0 & \pi & & & \\ \vdots & & & & & \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ b_1 \\ a_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} \int_{-\pi}^{\pi} f \cdot 1 dx \\ \int_{-\pi}^{\pi} f \cos(x) dx \\ \int_{-\pi}^{\pi} f \sin(x) dx \\ \int_{-\pi}^{\pi} f \cos(2x) dx \\ \vdots \end{bmatrix}$$



$$\begin{aligned} a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx \\ a_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(k\pi x) dx \\ b_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(k\pi x) dx \end{aligned} \quad k = 1, 2, \dots$$

Projection theory and the Fourier series

3.) Fourier series

Fourier series on arbitrary symmetric domain $[-p, p]$:

Basis functions:

$$\omega_k(x) = \begin{cases} \cos\left(\frac{k\pi x}{p}\right) \\ \sin\left(\frac{k\pi x}{p}\right) \\ 1 \end{cases} \quad l = 1, 2, \dots$$

Gramian:

$$G = \begin{bmatrix} 2p & 0 & 0 & 0 & 0 & \dots \\ 0 & p & 0 & 0 & 0 & \dots \\ 0 & 0 & p & & & \\ \vdots & & & & & \end{bmatrix}$$

$$\begin{bmatrix} 2p & 0 & 0 & 0 & 0 & \dots \\ 0 & p & 0 & 0 & 0 & \dots \\ 0 & 0 & p & & & \\ \vdots & & & & & \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ b_1 \\ a_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} \int_{-p}^p f \cdot 1 dx \\ \int_{-p}^p f \cos\left(\frac{\pi x}{p}\right) dx \\ \int_{-p}^p f \sin\left(\frac{\pi x}{p}\right) dx \\ \int_{-p}^p f \cos\left(\frac{2\pi x}{p}\right) dx \\ \vdots \end{bmatrix}$$



$$\begin{aligned} a_0 &= \frac{1}{2p} \int_{-p}^p f(x) dx \\ a_k &= \frac{1}{p} \int_{-p}^p f(x) \cos\left(\frac{k\pi x}{p}\right) dx \\ b_k &= \frac{1}{p} \int_{-p}^p f(x) \sin\left(\frac{k\pi x}{p}\right) dx \quad k = 1, 2, \dots \end{aligned}$$

Projection theory and the Fourier series

3.) Fourier series

Fourier series on arbitrary domain $[0, p]$:

Basis functions:

$$\omega_k(x) = \begin{cases} \cos\left(\frac{2l\pi x}{p}\right) \\ \sin\left(\frac{2l\pi x}{p}\right) \\ 1 \end{cases} \quad l = 1, 2, \dots$$

Gramian:

$$G = \begin{bmatrix} p & 0 & 0 & 0 & 0 & \dots \\ 0 & p/2 & 0 & 0 & 0 & \dots \\ 0 & 0 & p/2 & 0 & 0 & \dots \\ \vdots & & & & & \ddots \end{bmatrix}$$



$$a_0 = \frac{1}{p} \int_0^p f(x) dx$$

$$a_k = \frac{2}{p} \int_0^p f(x) \cos\left(\frac{2l\pi x}{p}\right) dx$$

$$b_k = \frac{2}{p} \int_0^p f(x) \sin\left(\frac{2l\pi x}{p}\right) dx \quad l = 1, 2, \dots$$

Projection theory and the Fourier series

3.) Fourier series

Let's find the a_k coefficients (the same without projection theorem)

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(k\pi x) + \sum_{k=1}^{\infty} b_k \sin(k\pi x)$$

Multiply both sides with $\cos(l\pi x)$ and integrate over the domain $[-1,1]$

$l = 0,1,2..$

$$\int_{-1}^1 f(x) \cos(l\pi x) dx =$$

$$\int_{-1}^1 \frac{a_0}{2} \cos(l\pi x) dx + \int_{-1}^1 \sum_{k=1}^{\infty} a_k \cos(k\pi x) \cos(l\pi x) dx + \int_{-1}^1 \sum_{k=1}^{\infty} b_k \sin(k\pi x) \cos(l\pi x) dx$$

$$\int_{-1}^1 f(x) \cos(l\pi x) dx =$$

$$\frac{a_0}{2} \int_{-1}^1 \cos(l\pi x) dx + \sum_{k=1}^{\infty} a_k \int_{-1}^1 \cos(k\pi x) \cos(l\pi x) dx + \sum_{k=1}^{\infty} b_k \int_{-1}^1 \sin(k\pi x) \cos(l\pi x) dx$$


Projection theory and the Fourier series

3.) Fourier series

Let's find the a_k coefficients (the same without projection theorem)

$$\int_{-1}^1 f(x) \cos(l\pi x) dx =$$
$$\frac{a_0}{2} \int_{-1}^1 \cos(l\pi x) dx + \sum_{k=1}^{\infty} a_k \int_{-1}^1 \cos(k\pi x) \cos(l\pi x) dx + \sum_{k=1}^{\infty} b_k \int_{-1}^1 \sin(k\pi x) \cos(l\pi x) dx$$

$\underbrace{\int_{-1}^1 \cos(l\pi x) dx}_{= \begin{cases} 2 & \text{for } l = 0 \\ 0 & \text{otherwise} \end{cases}}$ $\underbrace{\int_{-1}^1 \cos(k\pi x) \cos(l\pi x) dx}_{= \begin{cases} 0 & \text{for } k \neq l \\ 1 & \text{for } k = l \end{cases}}$ $\underbrace{\int_{-1}^1 \sin(k\pi x) \cos(l\pi x) dx}_{= 0}$

 $a_k = \int_{-1}^1 f(x) \cos(k\pi x) dx \quad k = 0, 1, 2, \dots$

Projection theory and the Fourier series

3.) Fourier series

Let's find the b_k coefficients (the same without projection theorem)

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(k\pi x) + \sum_{k=1}^{\infty} b_k \sin(k\pi x)$$

Multiply both sides with $\sin(l\pi x)$ and integrate over the domain $[-1,1]$

$$l = 1, 2, \dots$$

$$\int_{-1}^1 f(x) \sin(l\pi x) dx =$$

$$\int_{-1}^1 \frac{a_0}{2} \sin(l\pi x) dx + \int_{-1}^1 \sum_{k=1}^{\infty} a_k \cos(k\pi x) \sin(l\pi x) dx + \int_{-1}^1 \sum_{k=1}^{\infty} b_k \sin(k\pi x) \sin(l\pi x) dx$$

$$\int_{-1}^1 f(x) \sin(l\pi x) dx =$$

$$\frac{a_0}{2} \int_{-1}^1 \sin(l\pi x) dx + \sum_{k=1}^{\infty} a_k \int_{-1}^1 \cos(k\pi x) \sin(l\pi x) dx + \sum_{k=1}^{\infty} b_k \int_{-1}^1 \sin(k\pi x) \sin(l\pi x) dx$$

Projection theory and the Fourier series

3.) Fourier series

Let's find the b_k coefficients (the same without projection theorem)

$$\int_{-1}^1 f(x) \sin(l\pi x) dx =$$
$$\underbrace{\frac{a_0}{2} \int_{-1}^1 \sin(l\pi x) dx}_{= 0} + \sum_{k=1}^{\infty} a_k \underbrace{\int_{-1}^1 \cos(k\pi x) \sin(l\pi x) dx}_{= 0} + \sum_{k=1}^{\infty} b_k \underbrace{\int_{-1}^1 \sin(k\pi x) \sin(l\pi x) dx}_{= \begin{cases} 0 & \text{for } k \neq l \\ 1 & \text{for } k = l \end{cases}}$$



$$b_k = \int_{-1}^1 f(x) \sin(k\pi x) dx \quad k = 1, 2, \dots$$

Projection theory and the Fourier series

3.) Fourier series - - exercise

$$f(y) = -(y^2 - h^2/4)$$

$$f(y) = \sum_{k=1}^{\infty} A_k \sin\left(\frac{k\pi}{h} y\right)$$

- A_k coefficients?

$$A_k = \frac{2}{h} \int_0^h f(y) \sin\left(\frac{k\pi}{h} y\right) dy = \frac{2}{h} \int_0^h (-y(y-h)) \sin\left(\frac{k\pi}{h} y\right) dy$$

Projection theory and the Fourier series

3.) Fourier series - exercise

$$\begin{aligned} A_k &= \frac{-2}{h} \int_0^h (y(y-h)) \sin\left(\frac{k\pi}{h}y\right) dy = \frac{-2}{h} \int_0^h (y^2 - hy) \sin\left(\frac{k\pi}{h}y\right) dy = \\ & \frac{-2}{h} \frac{h}{k\pi} \left\{ \left[-(y^2 - hy) \cos\left(\frac{k\pi}{h}y\right) \right]_0^h + \int_0^h (2y - h) \cos\left(\frac{k\pi}{h}y\right) dy \right\} = \\ & \frac{-2}{h} \frac{h}{k\pi} \left\{ \left[-(y^2 - hy) \cos\left(\frac{k\pi}{h}y\right) \right]_0^h + \frac{h}{k\pi} \left[(2y - h) \sin\left(\frac{k\pi}{h}y\right) \right]_0^h - \frac{h}{k\pi} \int_0^h 2 \sin\left(\frac{k\pi}{h}y\right) dy \right\} \\ & \frac{-2}{h} \frac{h}{k\pi} \left\{ \underbrace{\left[-(y^2 - hy) \cos\left(\frac{k\pi}{h}y\right) \right]_0^h}_{=0} + \underbrace{\frac{h}{k\pi} \left[(2y - h) \sin\left(\frac{k\pi}{h}y\right) \right]_0^h}_{=0} + \underbrace{\left(\frac{h}{k\pi}\right)^2 \left[2 \cos\left(\frac{k\pi}{h}y\right) \right]_0^h}_{=2(-1)^k - 2} \right\} \end{aligned}$$

$$A_k = \begin{cases} 8 \frac{(h)^2}{(k\pi)^3} & \text{if } k \text{ is odd} \\ 0 & \text{if } k \text{ is even} \end{cases}$$