Technische
Universität
Braunschweig

Introduction to PDEs and Numerical Methods

Lecture 3:

Analytical solution of ODEs and PDEs

Dr. Noemi Friedman, 02.11.2016.

Overview of the course

- Introduction (definition of PDEs, classification, basic math, introductory examples of PDEs)

- Analytical solution of elementary PDEs (Fourier series/transform, separation of variables, Green's function)

- Numerical solutions of PDEs:

  - Finite difference method

  - Finite element method

Overview of this lecture

- Classification of PDEs revisited, some examples of PDEs again, boundary conditions√
- Solving linear systems – solving linear PDEs
- Eigenvalues, eigenfunctions, solving linear equations with spectral method – spectral method for PDEs
- Essential functional analysis, projection theory, Fourier series
- Separation of variables – analytical solution of the heat equation

Projection theory and the Fourier series
1.) Essential functional analysis - vector space

- **functional** is a function from a **vector space** into its underlying scalar field

- Vector spaces (V)

$$u, v \in V$$

  - addition of u and v ($z = u + v$) is defined such that $z \in V$
  - multiplication by scalars is defined such that $\alpha u \in V$

For addition and multiplication the following axioms are satisfied:

$\boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w}) = (\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w}$ **(associativity)**

$\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$ **(commutativity)**

There exists an element $\mathbf{0} \in V$, such that $\boldsymbol{v} + \mathbf{0} = \boldsymbol{v}$ for all $\boldsymbol{v} \in V$.

For every $\boldsymbol{v} \in V$, there exists an element $-\boldsymbol{v} \in V$ such that $\boldsymbol{v} + (-\boldsymbol{v}) = \mathbf{0}$

$a(b\boldsymbol{v}) = (ab)\boldsymbol{v}$

$1\boldsymbol{v} = \boldsymbol{v}$, where 1 denotes the multiplicative identity in F.

$a(\boldsymbol{u} + \boldsymbol{v}) = a\boldsymbol{u} + a\boldsymbol{v}$ **(distributivity)**

$(a + b)\boldsymbol{v} = a\boldsymbol{v} + b\boldsymbol{v}$ **(distributivity)**

with:

$$u, v, w \in V$$

$$a, b \in \mathbb{R}$$

Projection theory and the Fourier series
1.) Essential functional analysis - vector spaces, linear subspaces

Important vector spaces, examples:

- Euclidian n-space:

$$\mathbf{R}^n = \{(u_1, u_2, \ldots, u_n) \; : \; u_i \in \mathbf{R}, \, i = 1, 2, \ldots, n\}$$

- C[a, b] : set of all continuous, real-valued functions defined on the interval [a, b].
- C^1 [a, b]: set of all real-valued, continuously differentiable functions defined on the interval [a, b]. (A function is continuously differentiable if its derivative exists and is continuous.)
- C^k [a, b]: space of real-valued functions defined on [a, b] that have k continuous derivatives

Subspaces

linear subspaces $V_0 \in V, \;\; x, y \in V_0 \,, \;\; \mathrm{a}, \mathrm{b} \in \mathrm{R}, \;\; \mathrm{ax} + \mathrm{by} \in V_0$

(any linear combination of two elements of the subspace is also en element of the subspace)

examples:

$$C_D^2[a,b] = \left\{u \in C^2[a,b] \; : \; u(a) = u(b) = 0\right\}.$$ Dirichlet

$$C_N^2[a,b] = \left\{u \in C^2[a,b] \; : \; \frac{du}{dx}(a) = \frac{du}{dx}(b) = 0\right\}$$ Neumann

Projection theory and the Fourier series
1.) Essential functional analysis - norms

A vector space may be endowed with additional structures, such as a norm and inner product

Norm in Euclidian spaces: any mapping $g\colon V \to \mathbb{R}$, that satisfies:

1. The zero vector, **0**, has zero length; every other vector has a positive length.

$$g(\boldsymbol{x}) \geq 0, \;\; g(\boldsymbol{x}) = 0 \text{ only if } \boldsymbol{x} = \boldsymbol{0}$$

with the usual notation

$$\|\boldsymbol{x}\| \geq 0, \qquad \|\boldsymbol{x}\| = 0 \text{ only if } \boldsymbol{x} = \boldsymbol{0}$$

2. Multiplying a vector by a positive number changes its length without changing its direction.

$$g(a\boldsymbol{x}) = |a|g(\boldsymbol{x})$$

3. The triangle inequality holds. That is, taking norms as distances, the distance from point A through B to C is never shorter than going directly from A to C, or the shortest distance between any two points is a straight line.

$$g(\boldsymbol{x} + \boldsymbol{y}) \leq g(\boldsymbol{x}) + g(\boldsymbol{y})$$

Projection theory and the Fourier series
1.) Essential functional analysis - norms

Examples for norms in the Euclidien space

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\|\boldsymbol{x}\|_1 = |x_1| + |x_2| + |x_3|$$

$$\|\boldsymbol{x}\|_2 = \sqrt{{x_1}^2 + {x_2}^2 + {x_3}^2}$$

$$\|\boldsymbol{x}\|_p = ({x_1}^p + {x_2}^p + {x_3}^p)^{\frac{1}{p}}$$

$$\|\boldsymbol{x}\|_\infty = \max(|x_1|, |x_2|, |x_3|)$$

Projection theory and the Fourier series
1.) Essential functional analysis - norms

Norm in function spaces, equivavelently:

any mapping $g: V \rightarrow \mathbb{R}$, that satisfies:

1.

$$g(f(x)) \geq 0,\ \ g(f(x)) = 0 \text{ only if} f(x) = 0$$

with the usual notation

$$\|f(x)\| \geq 0, \quad \|f(x)\| = 0 \text{ only if} f(x) = 0$$

2.

$$g(af(x)) = |a| g(f(x))$$

3.

$$g(f(x) + h(x)) \leq g(f(x)) + g(h(x))$$

Examples:

$$\|f\|_2 = \sqrt{\int_0^1 f(x)^2 \mathrm{dx}} \qquad \|f\|_p = \left(\int_0^1 f(x)^p \mathrm{dx}\right)^{\frac{1}{p}}$$

$$\|f\|_\infty = \max_{x \in [0,1]} (f(x))$$

Projection theory and the Fourier series
1.) Essential functional analysis – inner products

Inner product space: vector space with an inner product

Inner product:
any mapping $g: V \times V \to \mathbb{R}$, that satisfies

1. Positivity

$$g(x,x) \geq 0,\ \ g(x,x) = 0 \text{ only if } x = 0$$

with the usual notation

$$\langle x,x \rangle \geq 0, \langle x,x \rangle = 0 \text{ only if } \ x = 0$$

or

$$(x,x) \geq 0, (x,x) = 0 \text{ only if } \ x = 0$$

2. Linearity

$$g(ax,y) = ag(x,y)$$
$$g(x+y,z) = g(x,z) + g(y,z)$$

3. Symmetry

$$g(x,y) = g(y,x)$$

Projection theory and the Fourier series
1.) Essential functional analysis – inner products

Examples

In Euclidian space ($\mathbb{R}^n$)

- Dot product

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \boldsymbol{x} \cdot \boldsymbol{y} =$$

$$\langle (x_1, \cdots, x_n), (y_1, \cdots, y_n) \rangle := x^{\mathrm{T}} y = \sum_{i=1}^{n} x_i y_i = x_1 y_1 + \cdots + x_n y_n,$$

- Hermitian form

$\langle \boldsymbol{x}, \boldsymbol{y} \rangle_A = \boldsymbol{xAy}$ if **A**: pos. def. symm.

In function spaces

- $\langle f, g \rangle = \int_0^1 f(x) g(x) dx$ $C_1[0,1]$
- $\langle f, g \rangle_{H1} = \int_0^1 f(x) g(x) + f'(x) g'(x) dx$

Projection theory and the Fourier series
2.) Projection theory

Approximation of a vector $\boldsymbol{u}$

we try to find the coefficients α_j of an approximating vector in the form:

$$\boldsymbol{u}_h = \sum_{j=1}^{n} \alpha_j \boldsymbol{v}_j$$

where

$\boldsymbol{v}_j$: known (linearly independent) vectors

$\boldsymbol{u}_h$: the approximation of the vector $\boldsymbol{u}$, which is in an n-dimensional space:

$$u_h \in V_h = \mathrm{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots \boldsymbol{v}_n\}$$

Approximation of a function $u(x)$

we try to find the coefficients α_j of a „proxi model” (ansatz function):

$$u_h\,(x) = \sum_{j=1}^{n} \alpha_j \omega_j\,(x)$$

where

$\omega_j(\mathrm{x})$: known (linearly independent) basis or ansatz functions

$u_h\,(x)$: the approximation of the solution $u(x)$, which is in an n-dimensional space:

$$u_h \in V_h = \mathrm{span}\{\omega_1, \omega_2, \ldots \omega_n\}$$

Projection theory and the Fourier series
2.) Projection theory

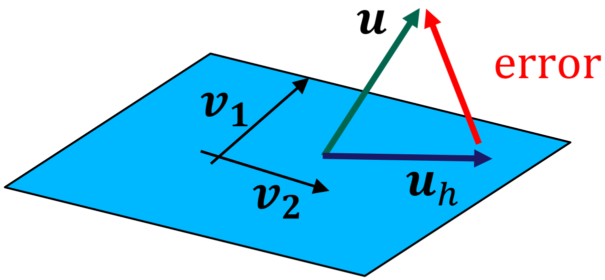
Approximation of a vector u	Approximation of a function $u(x)$

Our goal is to minimize the difference in between the solution and the approximation:

$$\text{error} = \|\boldsymbol{u} - \boldsymbol{u}_h\| < \|\boldsymbol{u} - \boldsymbol{z}\| \quad \forall \boldsymbol{z} \in V_h$$

$$\text{error} = \|u(\mathrm{x}) - u_h\,(\mathrm{x})\| < \|u(\mathrm{x}) - z(\mathrm{x})\| \quad \forall z(\mathrm{x}) \in V_h$$

The best approximation u_h to u from V_h is the one where the error is orthogonal to the space of V_h, that is to all possible $\mathrm{z} \in V_h$.
Instead of writing it for all z (as z is an n-dimensional space) we can write



$$\forall \boldsymbol{v_j}\ j = 1..n$$

$$\langle (u - u_h\,), v_j \rangle = 0 \quad j = 1..n$$

$$\forall \omega_j\ j = 1..n$$

$$\langle (u(\mathrm{x}) - u_h\,(\mathrm{x})), \omega_j(\mathrm{x}) \rangle = 0 \quad j = 1..n$$

Projection theory and the Fourier series
2.) Projection theory

Approximation of a vector $\boldsymbol{u}$

Approximation of a function $u(x)$

Plugging in the proxi model to the orthogonality condition we have:

$$\left\langle \left(u - \sum_{i=1}^{n} \alpha_i \boldsymbol{v}_i \right), \boldsymbol{v}_j \right\rangle = 0 \quad j = 1..n$$

$$\left\langle \left(u - \sum_{i=1}^{n} \alpha_i \omega_i \right), \omega_j \right\rangle = 0 \quad j = 1..n$$

Rearranging the equation we get:

$$\langle u, \boldsymbol{v}_j \rangle - \sum_{i=1}^{n} \alpha_i \langle \boldsymbol{v}_i, \boldsymbol{v}_j \rangle = 0 \quad j = 1..n$$

$$\sum_{i=1}^{n} \alpha_i \underbrace{\langle \boldsymbol{v}_i, \boldsymbol{v}_j \rangle}_{G_{ij}} = \underbrace{\langle u, \boldsymbol{v}_j \rangle}_{b_j} \quad j = 1..n$$

$$\langle u, \omega_j \rangle - \sum_{i=1}^{n} \alpha_i \langle \omega_i, \omega_j \rangle = 0 \quad j = 1..n$$

$$\sum_{i=1}^{n} \alpha_i \underbrace{\langle \omega_i, \omega_j \rangle}_{G_{ij}} = \underbrace{\langle u, \omega_j \rangle}_{b_j} \quad j = 1..n$$

$$\mathbf{G}\boldsymbol{\alpha} = \mathbf{b}$$

Projection theory and the Fourier series
3.) Fourier series

$f(y)$: some complicated function, we seek for a nicer interpretation in the form:

$$f(y) = \quad a_0 + \; a_1\cos(\pi x) + a_2\cos(2\pi x) + ..$$
$$b_1\sin(\pi x) + b_2\sin(2\pi x) + ..$$

- $a_k, \; b_k$ coefficients? Let's minimise the error of the approximation using the inner product: $\langle f, g\rangle = \int_{-1}^{1} \mathrm{f}(x)\, g(x) dx$ and the induced norm: $\sqrt{\langle f, f\rangle}$

First some important trigonometric identities:

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta \Longrightarrow \cos(\alpha)\cos(\beta) = \frac{1}{2}[\cos(\alpha+\beta) + \cos(\alpha-\beta)]$$

$$\Longrightarrow \sin(\alpha)\sin(\beta) = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$$

$$\Longrightarrow \cos^2(\alpha) = \frac{1}{2}[\cos(2\alpha) + 1]$$

$$\Longrightarrow \sin^2(\alpha) = \frac{1}{2}[1 - \cos(2\alpha)]$$

Projection theory and the Fourier series
3.) Fourier series

With the help of these identities,
we can proove, that:

$$\cos(\alpha)\cos(\beta) = \frac{1}{2}[\cos(\alpha+\beta) + \cos(\alpha-\beta)]$$

$$\sin(\alpha)\sin(\beta) = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$$

$$\cos^2(\alpha) = \frac{1}{2}[\cos(2\alpha) + 1]$$

$$\sin^2(\alpha) = \frac{1}{2}[1 - \cos(2\alpha)]$$

$$\int_{-\pi}^{\pi} \cos(kx)\sin(lx)dx = 0 \qquad (1)$$

$$\int_{-\pi}^{\pi} \cos(kx)\cos(lx)dx = \begin{cases} 0 & \text{for } k \neq l \\ \pi & \text{for } k = l \end{cases} \qquad (2)$$

$$\int_{-\pi}^{\pi} \sin(kx)\sin(lx)dx = \begin{cases} 0 & \text{for } k \neq l \\ \pi & \text{for } k = l \end{cases} \qquad (3)$$

Projection theory and the Fourier series
3.) Fourier series

Get the coefficients of the Fourier series of $f(x)$ with the projection theory

$$\sum_{i=1}^{n} \alpha_i \underbrace{\langle \omega_i, \omega_j \rangle}_{G_{ij}} = \underbrace{\langle f, \omega_j \rangle}_{b_j} \qquad j = 1..n$$

Coefficients: $\alpha_i = ?$

Basis functions:

$$\omega_k(x) = \begin{cases} \cos(\mathrm{l}x) \\ \sin(\mathrm{l}x) \\ 1 \end{cases} \qquad l = 1,2..$$

$$G = \begin{array}{c|cccccc} & 1 & \cos(x) & \sin(x) & \cos(2x) & \sin(2x) & \cdots \\ \hline 1 & \int_{-\pi}^{\pi} 1 \cdot 1 dx & \int_{-\pi}^{\pi} 1\cos(x)dx & \int_{-\pi}^{\pi} 1\sin(x)dx & \int_{-\pi}^{\pi} 1\cos(2x)dx & \int_{-\pi}^{\pi} 1\sin(2x)dx & \ldots \\ \cos(x) & \int_{-\pi}^{\pi} \cos(x)dx & \int_{-\pi}^{\pi} \cos(x)\sin(x)dx & \int_{-\pi}^{\pi} \cos(x)\sin(x)dx & \int_{-\pi}^{\pi} \cos(x)\cos(2x)\,dx & \int_{-\pi}^{\pi} \cos(x)\sin(2x)dx & \ldots \\ \sin(x) & \int_{-\pi}^{\pi} \sin(x)dx & \int_{-\pi}^{\pi} \sin(x)\cos(x)dx & \ddots & & & \\ \vdots & & \vdots & & & & \end{array}$$

Projection theory and the Fourier series
3.) Fourier series

$$G=\begin{bmatrix} \int_{-\pi}^{\pi} 1\cdot 1dx & \int_{-\pi}^{\pi} 1\cos(x)dx & \int_{-\pi}^{\pi} 1\sin(x)dx & \int_{-\pi}^{\pi} 1\cos(2x)dx & \int_{-\pi}^{\pi} 1\sin(2x)dx & \dots \\ \int_{-\pi}^{\pi} \cos(x)dx & \int_{-\pi}^{\pi} \cos(x)\sin(x)dx & \int_{-\pi}^{\pi} \cos(x)\sin(x)dx & \int_{-\pi}^{\pi} \cos(x)\cos(2x)\,dx & \int_{-\pi}^{\pi} \cos(x)\sin(2x)dx & \dots \\ \int_{-\pi}^{\pi} \sin(x)dx & \int_{-\pi}^{\pi} \cos(x)\sin(x)dx & \ddots & & & \\ & \vdots & & & & \end{bmatrix}$$

$$\int_{-\pi}^{\pi} 1\cdot 1dx = 2\pi \qquad \int_{-\pi}^{\pi} \cos(kx)\sin(lx)dx = 0 \qquad \int_{-\pi}^{\pi} \cos(kx)\cos(lx)dx = \begin{cases} 0 & \text{for } k\neq l \\ \pi & \text{for } k = l \end{cases}$$

$$\int_{-\pi}^{\pi} \sin(kx)\sin(lx)dx = \begin{cases} 0 & \text{for } k\neq l \\ \pi & \text{for } k = l \end{cases} \qquad \int_{-\pi}^{\pi} \sin(\mathrm{l}x)\,dx = 0 \qquad \int_{-\pi}^{\pi} \cos(\mathrm{l}x)\,dx = \begin{cases} 0 & \text{for } l = 1,2,3.. \\ 2\pi & \text{for } l = 0 \end{cases}$$

$$G=\begin{bmatrix} 2\pi & 0 & 0 & 0 & 0 & \dots \\ 0 & \pi & 0 & 0 & 0 & \dots \\ 0 & 0 & \pi & & & \\ \vdots & & & & & \end{bmatrix}$$

$$\mathbf{G}\boldsymbol{\alpha} = \mathbf{b}$$

$$\mathbf{b} = \langle f, \omega_j \rangle = \begin{bmatrix} \int_{-\pi}^{\pi} f\cdot 1dx \\ \int_{-\pi}^{\pi} f\cos(x)dx \\ \int_{-\pi}^{\pi} f\sin(x)dx \\ \int_{-\pi}^{\pi} f\cos(1x)dx \\ \vdots \end{bmatrix}$$

Projection theory and the Fourier series
3.) Fourier series

$$\begin{bmatrix} 2\pi & 0 & 0 & 0 & 0 & \ldots \\ 0 & \pi & 0 & 0 & 0 & \ldots \\ 0 & 0 & \pi & & & \\ \vdots & & & & & \\ & & & & & \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ b_1 \\ a_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} \int_{-\pi}^{\pi} f \cdot 1 dx \\ \int_{-\pi}^{\pi} f \cos(x) dx \\ \int_{-\pi}^{\pi} f \sin(x) dx \\ \int_{-\pi}^{\pi} f \cos(1x) dx \\ \vdots \end{bmatrix}$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(\mathrm{l}\pi x)\, dx$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(\mathrm{l}\pi x)\, dx$$

$$l = 1{,}2..$$

Projection theory and the Fourier series
3.) Fourier series

Fourier series on arbitrary symmetric domain $[-p, p]$:

Basis functions:

$$\omega_k(x) = \begin{cases} \cos\left(\frac{\mathrm{l}\pi x}{p}\right) \\ \sin\left(\frac{\mathrm{l}\pi x}{p}\right) \\ 1 \end{cases} \quad l = 1,2..$$

Gramian:

$$G = \begin{bmatrix} 2p & 0 & 0 & 0 & 0 & \dots \\ 0 & p & 0 & 0 & 0 & \dots \\ 0 & 0 & p & & & \\ \vdots & & & & & \\ & & & & & \end{bmatrix}$$

$$\begin{bmatrix} 2p & 0 & 0 & 0 & 0 & \dots \\ 0 & p & 0 & 0 & 0 & \dots \\ 0 & 0 & p & & & \\ \vdots & & & & & \\ & & & & & \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ b_1 \\ a_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} \int_{-p}^{p} f \cdot 1 dx \\ \int_{-p}^{p} f \cos\left(\frac{\pi x}{p}\right) dx \\ \int_{-p}^{p} f \sin\left(\frac{\pi x}{p}\right) dx \\ \int_{-p}^{p} f \cos\left(\frac{2\pi x}{p} x\right) dx \\ \vdots \end{bmatrix}$$

$$a_0 = \frac{1}{2p} \int_{-p}^{p} f(x) dx$$

$$a_k = \frac{1}{p} \int_{-p}^{p} f(x) \cos\left(\frac{\mathrm{l}\pi x}{p}\right) dx$$

$$b_k = \frac{1}{p} \int_{-p}^{p} f(x) \sin\left(\frac{\mathrm{l}\pi x}{p}\right) dx$$

$$l = 1,2..$$

Projection theory and the Fourier series
3.) Fourier series

Fourier series on arbitrary domain $[0, p]$:

Basis functions:

$$\omega_k(x) = \begin{cases} \cos\left(\frac{2l\pi x}{p}\right) \\ \sin\left(\frac{2l\pi x}{p}\right) \\ 1 \end{cases} \quad l = 1,2..$$

Gramian:

$$G = \begin{bmatrix} p & 0 & 0 & 0 & 0 & \dots \\ 0 & p/2 & 0 & 0 & 0 & \dots \\ 0 & 0 & p/2 & & & \\ & \vdots & & & & \\ & & & & & \end{bmatrix}$$

$$\begin{bmatrix} p & 0 & 0 & 0 & 0 & \dots \\ 0 & p/2 & 0 & 0 & 0 & \dots \\ 0 & 0 & p/2 & & & \\ & \vdots & & & & \\ & & & & & \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ b_1 \\ a_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} \int_{-p}^{p} f \cdot 1 dx \\ \int_{-p}^{p} f \cos(x) dx \\ \int_{-p}^{p} f \sin(x) dx \\ \int_{-p}^{p} f \cos(1x) dx \\ \vdots \end{bmatrix}$$

$$a_0 = \frac{1}{p} \int_0^p f(x) dx$$

$$a_k = \frac{2}{p} \int_0^p f(x) \cos\left(\frac{2l\pi x}{p}\right) dx$$

$$b_k = \frac{2}{p} \int_0^p f(x) \sin\left(\frac{2l\pi x}{p}\right) dx$$

$$l = 1,2..$$

Projection theory and the Fourier series
3.) Fourier series

Let's find the a_k coefficients (the same without projection theorem)

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(\mathrm{k}\pi x) + \sum_{k=1}^{\infty} b_k \sin(\mathrm{k}\pi x)$$

Multiply both sides with $\cos(\mathrm{l}\pi x)$ and integrate over the domain [-1,1]

$$l = 0{,}1{,}2..$$

$$\int_{-1}^{1} f(x) \cos(\mathrm{l}\pi x)\, dx =$$

$$\int_{-1}^{1} \frac{a_0}{2} \cos(\mathrm{l}\pi x)\, dx + \int_{-1}^{1} \sum_{k=1}^{\infty} a_k \cos(\mathrm{k}\pi x) \cos(\mathrm{l}\pi x)\, dx + \int_{-1}^{1} \sum_{k=1}^{\infty} b_k \sin(\mathrm{k}\pi x) \cos(\mathrm{l}\pi x)\, dx$$

$$\int_{-1}^{1} f(x) \cos(\mathrm{l}\pi x)\, dx =$$

$$\frac{a_0}{2} \int_{-1}^{1} \cos(\mathrm{l}\pi x)\, dx + \sum_{k=1}^{\infty} a_k \int_{-1}^{1} \cos(\mathrm{k}\pi x) \cos(\mathrm{l}\pi x)\, dx + \sum_{k=1}^{\infty} b_k \int_{-1}^{1} \sin(\mathrm{k}\pi x) \cos(\mathrm{l}\pi x)\, dx$$

Projection theory and the Fourier series
3.) Fourier series

Let’s find the a_k coefficients (the same without projection theorem)

$$\int_{-1}^{1} f(x)\,\cos(\mathrm{l}\pi x)\,dx =$$

$$\frac{a_0}{2}\underbrace{\int_{-1}^{1}\cos(\mathrm{l}\pi x)\,dx}_{=\begin{cases}2 \ \ \text{for } l=0\\ 0 \ \ \text{otherwise}\end{cases}} + \sum_{k=1}^{\infty} a_k \underbrace{\int_{-1}^{1}\cos(\mathrm{k}\pi x)\,\cos(\mathrm{l}\pi x)\,dx}_{=\begin{cases}0 \ \ \text{for } k\neq l\\ 1 \ \ \text{for } k=l\end{cases}} + \sum_{k=1}^{\infty} b_k \underbrace{\int_{-1}^{1}\sin(\mathrm{k}\pi x)\,\cos(\mathrm{l}\pi x)\,dx}_{=0}$$

$$\boxed{a_k = \int_{-1}^{1} f(x)\,\cos(\mathrm{k}\pi x)\,dx \qquad k = 0,1,2..}$$

Projection theory and the Fourier series
3.) Fourier series

Let's find the b_k coefficients (the same without projection theorem)

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(\mathrm{k}\pi x) + \sum_{k=1}^{\infty} b_k \sin(\mathrm{k}\pi x)$$

Multiply both sides with $\sin(\mathrm{l}\pi x)$ and integrate over the domain [-1,1]

$l = 1{,}2..$

$$\int_{-1}^{1} f(x) \sin(\mathrm{l}\pi x)\, dx =$$

$$\int_{-1}^{1} \frac{a_0}{2} \sin(\mathrm{l}\pi x)\, dx + \int_{-1}^{1} \sum_{k=1}^{\infty} a_k \cos(\mathrm{k}\pi x) \sin(\mathrm{l}\pi x)\, dx + \int_{-1}^{1} \sum_{k=1}^{\infty} b_k \sin(\mathrm{k}\pi x) \sin(\mathrm{l}\pi x)\, dx$$

$$\int_{-1}^{1} f(x) \sin(\mathrm{l}\pi x)\, dx =$$

$$\frac{a_0}{2} \int_{-1}^{1} \sin(\mathrm{l}\pi x)\, dx + \sum_{k=1}^{\infty} a_k \int_{-1}^{1} \cos(\mathrm{k}\pi x) \sin(\mathrm{l}\pi x)\, dx + \sum_{k=1}^{\infty} b_k \int_{-1}^{1} \sin(\mathrm{k}\pi x) \sin(\mathrm{l}\pi x)\, dx$$

Projection theory and the Fourier series
3.) Fourier series

Let’s find the b_k coefficients (the same without projection theorem)

$$\int_{-1}^{1} f(x)\,\sin(\mathrm{l}\pi x)\,dx =$$

$$\underbrace{\frac{a_0}{2}\int_{-1}^{1} \sin(\mathrm{l}\pi x)\,dx}_{=0} + \sum_{k=1}^{\infty} a_k \underbrace{\int_{-1}^{1} \cos(\mathrm{k}\pi x)\,\sin(\mathrm{l}\pi x)\,dx}_{=0} + \sum_{k=1}^{\infty} b_k \underbrace{\int_{-1}^{1} \sin(\mathrm{k}\pi x)\,\sin(\mathrm{l}\pi x)\,dx}_{=\begin{cases}0 & \text{for } k \neq l \\ 1 & \text{for } k = l\end{cases}}$$

$$\boxed{b_k = \int_{-1}^{1} f(x)\,\sin(\mathrm{k}\pi x)\,dx \qquad k = 1,2..}$$

Projection theory and the Fourier series
3.) Fourier series - - exercise

$f(y) = -(y^2 - h^2/4)$

$$f(y) = \sum_{k=1}^{\infty} A_k \sin\left(\frac{\mathrm{k\pi}}{h} y\right)$$

- A_k coefficients?

$$A_k = \frac{2}{h}\int_0^h f(y)\sin\left(\frac{\mathrm{k\pi}}{h}y\right)\mathrm{dy} = \frac{2}{h}\int_0^h (-y(y-h))\sin\left(\frac{\mathrm{k\pi}}{h}y\right)\mathrm{dy}$$

Projection theory and the Fourier series
3.) Fourier series - exercise

$$A_k = \frac{-2}{h}\int_0^h (y(y-h))\sin\left(\frac{\mathrm{k\pi}}{h}y\right)\mathrm{dy} = \frac{-2}{h}\int_0^h (y^2-hy)\sin\left(\frac{\mathrm{k\pi}}{h}y\right)\mathrm{dy} =$$

$$\frac{-2}{h}\frac{h}{\mathrm{k\pi}}\left\{\left[-(y^2-hy)\cos\left(\frac{\mathrm{k\pi}}{h}y\right)\right]_0^{\mathrm{h}} + \int_0^h (2y-h)\cos\left(\frac{\mathrm{k\pi}}{h}y\right)\mathrm{dy}\right\} =$$

$$\frac{-2}{h}\frac{h}{\mathrm{k\pi}}\left\{\left[-(y^2-hy)\cos\left(\frac{\mathrm{k\pi}}{h}y\right)\right]_0^{\mathrm{h}} + \frac{h}{\mathrm{k\pi}}\left[(2y-h)\sin\left(\frac{\mathrm{k\pi}}{h}y\right)\right]_0^{\mathrm{h}} - \frac{h}{\mathrm{k\pi}}\int_0^h 2\sin\left(\frac{\mathrm{k\pi}}{h}y\right)\mathrm{dy}\right\}$$

$$\frac{-2}{h}\frac{h}{\mathrm{k\pi}}\left\{\underbrace{\left[-(y^2-hy)\cos\left(\frac{\mathrm{k\pi}}{h}y\right)\right]_0^{\mathrm{h}}}_{=0} + \frac{h}{\mathrm{k\pi}}\underbrace{\left[(2y-h)\sin\left(\frac{\mathrm{k\pi}}{h}y\right)\right]_0^{\mathrm{h}}}_{=0} + \left(\frac{h}{\mathrm{k\pi}}\right)^2\underbrace{\left[2\cos\left(\frac{\mathrm{k\pi}}{h}y\right)\right]_0^{\mathrm{h}}}_{=2(-1)^k-2}\right\}$$

$$A_k = \begin{cases} 8\dfrac{(h)^2}{(\mathrm{k\pi})^3} & if\ k\ is\ odd \\ 0 & if\ k\ is\ even \end{cases}$$