



Technische
Universität
Braunschweig



Introduction to PDEs and Numerical Methods

Lecture 2:

Analytical solution of ODEs and PDEs

Dr. Noemi Friedman, 25.10. 2016.

Overview of the course

- Introduction (definition of PDEs, **classification**, basic math, **modelling: introductory examples of PDEs**)
- **Analytical solution of elementary PDEs** (Fourier series/transform, **separation of variables**, Green's function)
- Numerical solutions of PDEs:
 - Finite difference method
 - Finite element method

Overview of this lecture

- Classification of PDEs, some examples of PDEs, boundary conditions
- Solving linear systems – solving linear PDEs
- Eigenvalues, eigenfunctions, solving linear equations with spectral method – spectral method for PDEs
- Fourier series
- Separation of variables – analytical solution of the heat equation

Classification of PDEs

- Constant/variable coefficients
- Stationary/instationary (not time dependent/time dependent)
- Linear/nonlinear
linearity condition: $L(\alpha u + \beta v) = \alpha L(u) + \beta L(v)$
- order
order of the highest derivative
- homogeneous/inhomogeneous
inhomogeneous: additive terms which do not depend on unknown function
homogeneous: $u = 0$ is a solution of the equation
- elliptic/parabolic/hyperbolic (only for second order PDEs)
 $Au_{xx} + 2Bu_{xy} + Cu_{yy} + \text{lower order derivatives} = 0$
 - $AC - B^2 = 0$ parabolic
 - $AC - B^2 < 0$ hyperbolic
 - $AC - B^2 > 0$ elliptic

Classification of PDEs, examples of PDEs

	Wave equation $u_{tt} - c^2 u_{xx} = 0$	Laplace equation $u_{xx} + u_{yy} = 0$	Heat equation $u_t - u_{xx} = 0$
Order	2	2	2
Constant coefficient?	yes	yes	yes
Linear?	yes	yes	yes
Homogenous?	yes	yes	yes
Class	$A=1, B=0, C=-c^2$ $AC - B^2 = -c^2$ Hyperbolic	$A=1, B=0, C=1$ $AC - B^2 = 1$ Elliptic	$A=-1, B=0, C=0$ $AC - B^2 = 0$ Parabolic

Classification of PDEs, examples of PDEs

$$L_2(u(t, x)) = \frac{\partial^2 u}{\partial t^2} - \alpha \frac{\partial^2 u}{\partial x^2}$$

The L_2 operator is hyperbolic if $\alpha > 0$

Linearity:

$$\begin{aligned} L_2(\beta u_1 + \gamma u_2) &= \\ &= \frac{\partial^2(\beta u_1 + \gamma u_2)}{\partial t^2} - \alpha \frac{\partial^2(\beta u_1 + \gamma u_2)}{\partial x^2} = \\ &= \beta \frac{\partial^2(u_1)}{\partial t^2} + \gamma \frac{\partial^2(u_2)}{\partial t^2} - \alpha \left(\beta \frac{\partial^2(u_1)}{\partial x^2} - \gamma \frac{\partial^2(u_2)}{\partial x^2} \right) = \\ &= \beta \left(\frac{\partial^2(u_1)}{\partial t^2} - \alpha \frac{\partial^2(u_1)}{\partial x^2} \right) + \gamma \left(\frac{\partial^2(u_2)}{\partial t^2} - \alpha \frac{\partial^2(u_2)}{\partial x^2} \right) = \\ &= \beta L_2(u_1) + \gamma L_2(u_2) \end{aligned}$$

Classification of PDEs, examples of PDEs

$$\frac{\partial^4 u}{\partial x^4} + 2 \frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} = -\frac{q}{D} \quad \text{order: 4; linear, stationary}$$

$$\frac{\partial u}{\partial t} - 4u \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} = 0 \quad \text{order: 2; nonlinear, instationary}$$

$$\frac{\partial^2 u}{\partial t^2} - \left(\frac{\partial^2 u}{\partial x^2}\right)^2 = f \quad \text{order: 2; nonlinear, instationary}$$

Introductory example: heat flow in a bar

- Basic assumptions:
- Uniform cross-section
- Temperature varies only in the longitudinal direction
- Relationship between heat energy and temperature is linear

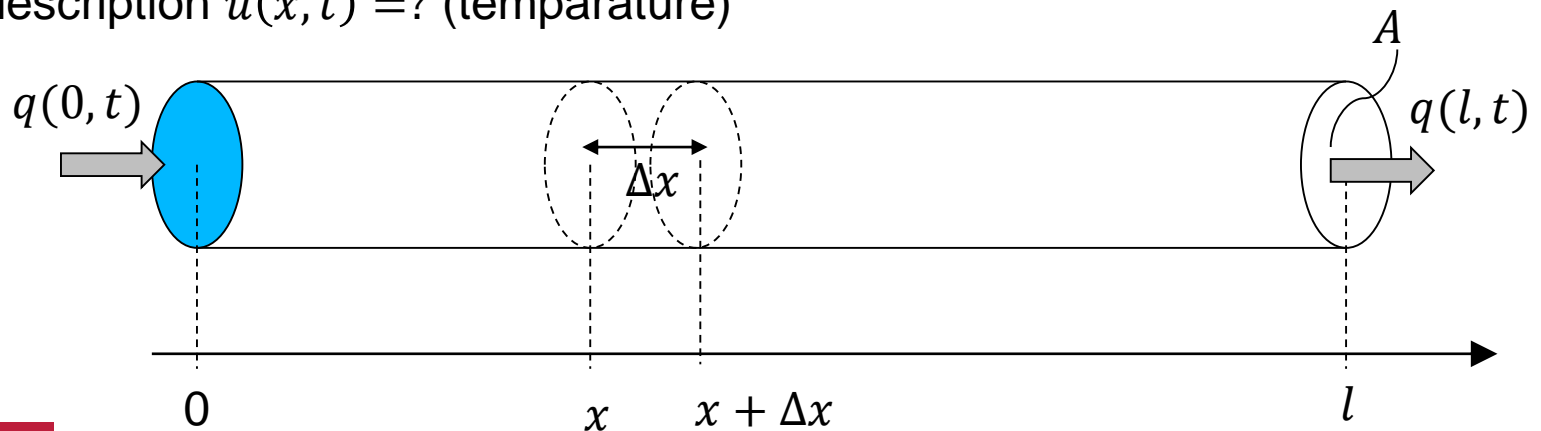
$c \left[\frac{J}{gK} \right]$: specific heat capacity \longleftrightarrow Energy [J]

c J energy is required to raise the temperature by 1K of 1g material

- Homogenous material properties along the bar (ρ and c are constants along the bar)

$\rho \left[\frac{g}{m^3} \right]$: density of the material of the bar

- Problem description $u(x, t) = ?$ (temperature)

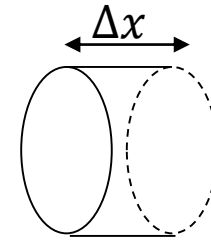


Introductory example: heat flow in a bar

Total energy in the bar section of length Δx is:

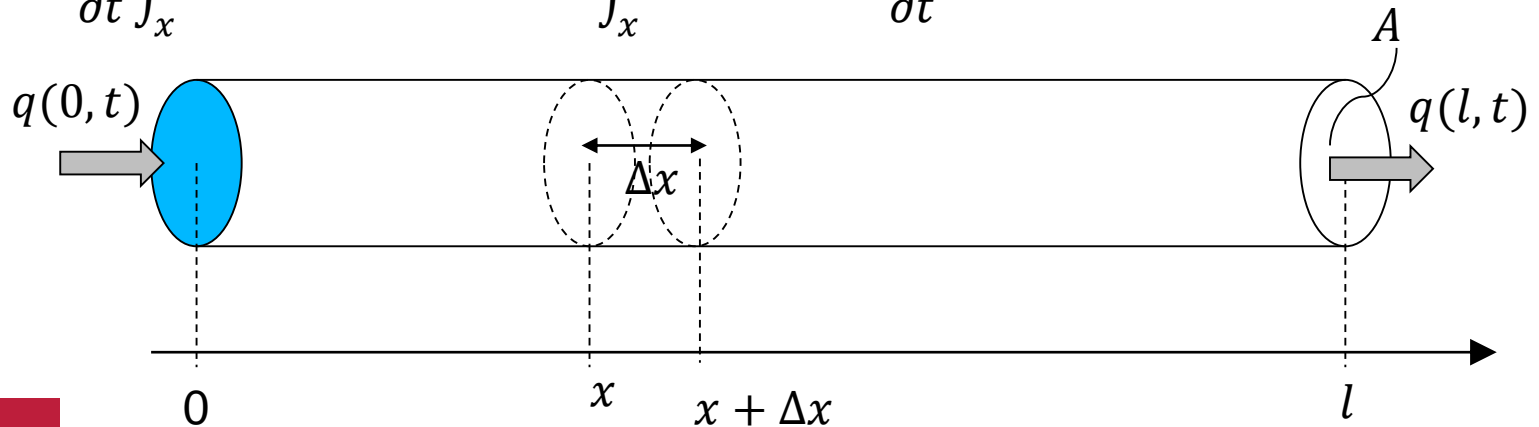
$$E_{tot} = E_0 + \int_x^{x+\Delta x} A\rho c(u(s,t) - T_0)ds =$$

$$E_0 + \int_x^{x+\Delta x} A\rho c u(s,t)ds - \underbrace{\int_x^{x+\Delta x} A\rho c T_0 ds}_{A\rho c T_0 \Delta x}$$



But I'm only interested in:

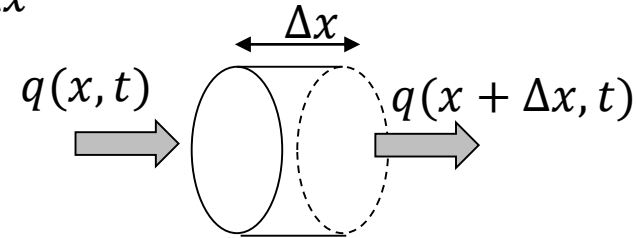
$$\frac{\partial}{\partial t} \int_x^{x+\Delta x} A\rho c u(s,t)ds = \int_x^{x+\Delta x} A\rho c \frac{\partial u(s,t)}{\partial t} ds$$



Introductory example: heat flow in a bar

Change of heat energy by time in the section of length Δx

$$\frac{\partial}{\partial t} \int_x^{x+\Delta x} A\rho c u(s, t) ds = \int_x^{x+\Delta x} A\rho c \frac{\partial u(s, t)}{\partial t} ds$$



Change of energy by time in the section of length Δx from heat flux:

$q(x, t) \left[\frac{J}{m^2 s} \right]$: heat flux

$$Aq(x + \Delta x, t) - Aq(x, t) = \int_x^{x+\Delta x} A \frac{\partial q(s, t)}{\partial x} ds$$

Fundamental theorem of calculus:

$$\int_a^b f(x) dx = F(b) - F(a).$$

Conservation of energy:

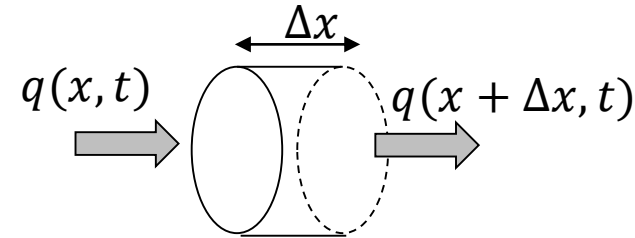
$$\int_x^{x+\Delta x} A \frac{\partial q(s, t)}{\partial x} ds = - \int_x^{x+\Delta x} A\rho c \frac{\partial u(s, t)}{\partial t} ds$$

$$\Rightarrow \int_x^{x+\Delta x} A \frac{\partial q(s, t)}{\partial x} + A\rho c \frac{\partial u(s, t)}{\partial t} ds = 0$$

Introductory example: heat flow in a bar

$$\int_x^{x+\Delta x} A \frac{\partial q(s, t)}{\partial x} + A\rho c \frac{\partial u(s, t)}{\partial t} ds = 0$$

$$\frac{\partial q(x, t)}{\partial x} + \rho c \frac{\partial u(x, t)}{\partial t} = 0 \quad 0 < x < l$$



$\frac{\partial u(x, t)}{\partial x}$: change of temperature with increasing x

Assumption: Fourier's law of heat conduction:

$$q(x, t) = -\kappa \frac{\partial u(x, t)}{\partial x}$$

κ : thermal conductivity

(for homogenous material properties)

Heat equation

$$\rho c \frac{\partial u(x, t)}{\partial t} - \kappa \frac{\partial^2 u(x, t)}{\partial x^2} = 0$$

$$\rho(x)c(x) \frac{\partial u(x, t)}{\partial t} - \frac{\partial}{\partial x} \kappa(x) \frac{\partial u(x, t)}{\partial x} = 0 \quad (\text{for inhomogenous material properties})$$

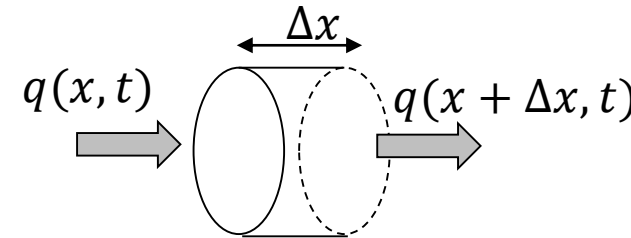
Introductory example: heat flow in a bar

The heat equation with source or sink (inhomogenous heat equation)

$f(s, t)$: heat energy per unit time, per unit volume

$$\int_x^{x+\Delta x} A \frac{\partial q(s, t)}{\partial x} + A\rho c \frac{\partial u(s, t)}{\partial t} ds = \int_x^{x+\Delta x} Af(s, t) ds$$

$$\frac{\partial q(x, t)}{\partial x} + \rho c \frac{\partial u(x, t)}{\partial t} = f(x, t) \quad 0 < x < l$$

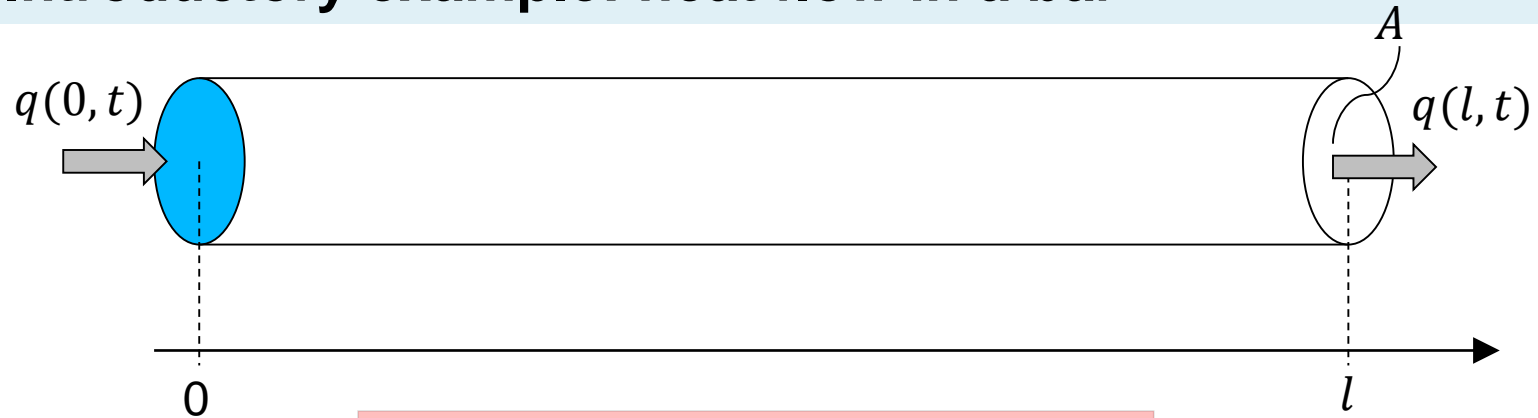


Heat equation

$$\rho c \frac{\partial u(x, t)}{\partial t} - \kappa \frac{\partial^2 u(x, t)}{\partial x^2} = f(x, t) \quad (\text{for homogenous material properties})$$

$$\rho(x)c(x) \frac{\partial u(x, t)}{\partial t} - \frac{\partial}{\partial x} \kappa(x) \frac{\partial u(x, t)}{\partial x} = f(x, t) \quad (\text{for inhomogenous material properties})$$

Introductory example: heat flow in a bar



$$\rho c \frac{\partial u(x, t)}{\partial t} - \kappa \frac{\partial^2 u(x, t)}{\partial x^2} = f(x, t)$$

Boundary conditions:

a) Perfect isolation at the end (flux across the boundaries is zero):

$$-\kappa \frac{\partial u(0, t)}{\partial x} = 0 \quad -\kappa \frac{\partial u(l, t)}{\partial x} = 0 \quad \forall t$$

b) Perfect thermal contact:

$$u(0, t) = 0 \quad u(l, t) = 0 \quad \forall t$$

Examples of PDEs and their classification

Transport equation: $u_t + cu_x = 0$

Heat equation: $\rho c u_t - \kappa u_{xx} = f$ $\rho(x)c(x)u_t - \frac{\partial}{\partial x}(\kappa(x)u_x) = f \Rightarrow \rho c u_t - \kappa \Delta u = f$
 (Diffusion/membrane equation/
 electrical conduction problem \rightarrow temperature: electric potential, heat flux: electric current)

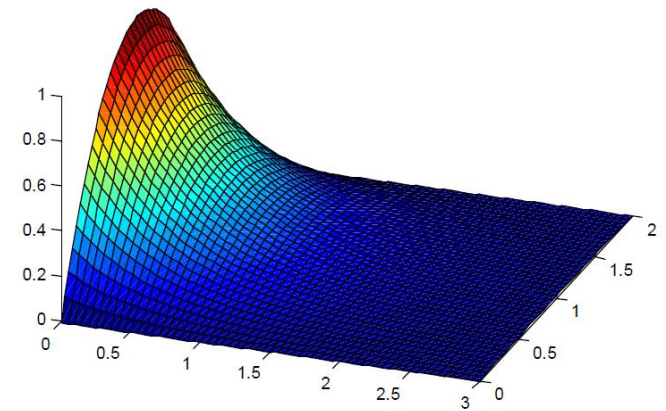
Steady state of the heat equation: $\lim_{t \rightarrow \infty} \frac{\partial}{\partial t} u = 0$

$$-\kappa u_{xx} = f \quad \longrightarrow \quad -\kappa \Delta u = f \quad -\kappa u_{xx} = 0 \quad \longrightarrow \quad -\Delta u = 0 \text{ (Laplace equation)}$$

(Poisson's equation)

Hanging bar: $\rho u_{tt} - \kappa u_{xx} = f$
 (the **wave equation** : vibrating string)

Steady state of the hanging bar: $-\kappa u_{xx} = f$



Examples of PDEs and their classification

Gauss's law: $\nabla \cdot E = \frac{\rho}{\varepsilon_0}$ ε_0 : permittivity of free space, E electrical field

Gauss's law for magnetism: $\nabla \cdot B = 0$ B : magnetic field

Faraday' law of induction: $\nabla \times B = -B_t$ B_t : magnetic flux

Burger's equation: $u_t + uu_x = \nu u_{xx}$ ν viscosity/diffusion coefficient
(fluid mechanics, nonlinear acoustics, gas dynamics, traffic flow)

Inviscid Burger's equation: $u_t + uu_x = 0$

Plate equation: $u_{xxxx} + u_{xxyy} + u_{yyyy} = 0$

Analytical solution of the 2D Laplace equation

- (a) The wave equation: $u_{tt} - c^2 u_{xx} = 0$, *hyperbolic*,
(b) The heat equation: $u_t - \gamma u_{xx} = 0$, *parabolic*,
(c) Laplace's equation: $u_{xx} + u_{yy} = 0$, *elliptic*.

$$\begin{aligned}\Delta u = u_{xx} + u_{yy} &= 0, & (x, y) \in \Omega \\ u(0, y) = u(1, y) &= 0, & y \in [0, 1] \\ u(x, 0) &= 0, & x \in [0, 1] \\ u(x, 1) &= -x^2 + x, & x \in (0, 1)\end{aligned}$$

Analytical solution of the 2D Laplace equation

$$\Delta u = u_{xx} + u_{yy} = 0, \quad (x, y) \in \Omega$$

- Solve by separation of variables
- ansatz + separation:

$$u(x, y) = v(x) w(y)$$

$$\frac{\partial^2 u}{\partial x^2} = v''(x) w(y), \quad \frac{\partial^2 u}{\partial y^2} = v(x) w''(y)$$

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = v''(x) w(y) + v(x) w''(y) = 0$$

$$v''(x) w(y) = -v(x) w''(y)$$

$$\frac{v''(x)}{v(x)} = -\frac{w''(y)}{w(y)} = \lambda$$

Analytical solution of the 2D Laplace equation

$$\frac{v''(x)}{v(x)} = -\frac{w''(y)}{w(y)} = \lambda \quad \longrightarrow \quad \text{separation constant}$$

$$v'' - \lambda v = 0,$$

$$w'' + \lambda w = 0$$

eigenvalue-eigenfunction problem: $Lu = \lambda u$

$$w'' + \lambda w = 0$$

a) Let's assume $\lambda = 0$

$$w''(y) = 0$$

Any linear equation would satisfy this equation:

$$w(y) = C_1 y + C_2$$

b) Let's assume $\lambda \neq 0$

Assuming the solution in the form:

$$w(y) = e^{\sigma y}$$

$$\sigma^2 e^{\sigma y} + \lambda e^{\sigma y} = 0 \quad (\text{characteristic equation})$$

$$\sigma^2 = -\lambda \quad \longrightarrow \quad \sigma = \pm \sqrt{-\lambda}$$

The solution is the linear combination of the two possible solutions

$$w(y) = C_3 e^{\sqrt{-\lambda}y} + C_4 e^{-\sqrt{-\lambda}y}$$

C_3, C_4 are arbitrary constants

Eigenvalue problem – Poisson equation with hom. D.B.C

Let's assume $\lambda > 0$

$$\sigma = \pm i\sqrt{\lambda}$$

The solution:

$$w(y) = C_3 e^{i\sqrt{\lambda}y} + C_4 e^{-i\sqrt{\lambda}y} =$$

$$\begin{aligned} & C_3 \{ \cos(\sqrt{\lambda}y) + i \sin(\sqrt{\lambda}y) \} + C_4 \{ \cos(\sqrt{\lambda}y) - i \sin(\sqrt{\lambda}y) \} = \\ & \underbrace{(C_3 + C_4)}_{= A} \cos(\sqrt{\lambda}y) + \underbrace{(C_3 - C_4)i}_{= B} \sin(\sqrt{\lambda}y) \end{aligned}$$

where A, B are new constants with

$$C_3 + C_4 = A \qquad (C_3 - C_4)i = B$$

$$w(y) = A \cos(\sqrt{\lambda}y) + B \sin(\sqrt{\lambda}y)$$

Introducing : $\lambda = \omega^2$

$$w(y) = A \cos(\omega y) + B \sin(\omega y)$$

Let's assume $\lambda < 0$

$$w(y) = C_3 e^{\sqrt{-\lambda}y} + C_4 e^{-\sqrt{-\lambda}y}$$

Introducing : $\lambda = -\omega^2$

$$w(y) = C_3 e^{\omega y} + C_4 e^{-\omega y}$$

λ : separation constant,
eigenvalue of $L = \frac{\partial}{\partial x^2}$

ω : **eigenfrequency** of $L = \frac{\partial}{\partial x^2}$

Analytical solution of the 2D Laplace equation

$$\frac{v''(x)}{v(x)} = -\frac{w''(y)}{w(y)} = \lambda \quad \longrightarrow \quad \text{separation constant}$$

$$\swarrow \qquad \searrow$$

$$v'' - \lambda v = 0, \qquad w'' + \lambda w = 0$$

λ	$v(x)$	$w(y)$	$u(x, y) = v(x) w(y)$
$\lambda = -\omega^2 < 0$	$\cos \omega x, \sin \omega x$	$e^{-\omega y}, e^{\omega y},$	$e^{\omega y} \cos \omega x, e^{\omega y} \sin \omega x,$ $e^{-\omega y} \cos \omega x, e^{-\omega y} \sin \omega x$
$\lambda = 0$	$1, x$	$1, y$	$1, x, y, xy$
$\lambda = \omega^2 > 0$	$e^{-\omega x}, e^{\omega x}$	$\cos \omega y, \sin \omega y$	$e^{\omega x} \cos \omega y, e^{\omega x} \sin \omega y,$ $e^{-\omega x} \cos \omega y, e^{-\omega x} \sin \omega y$

Analytical solution of the 2D Laplace equation

$$v'' - \lambda v = 0,$$

$$w'' + \lambda w = 0$$

λ	$v(x)$	$w(y)$	$u(x, y) = v(x) w(y)$
$\lambda = -\omega^2 < 0$	$\cos \omega x, \sin \omega x$	$e^{-\omega y}, e^{\omega y},$	$e^{\omega y} \cos \omega x, e^{\omega y} \sin \omega x,$ $e^{-\omega y} \cos \omega x, e^{-\omega y} \sin \omega x$
$\lambda = 0$	$1, x$	$1, y$	$1, x, y, xy$
$\lambda = \omega^2 > 0$	$e^{-\omega x}, e^{\omega x}$	$\cos \omega y, \sin \omega y$	$e^{\omega x} \cos \omega y, e^{\omega x} \sin \omega y,$ $e^{-\omega x} \cos \omega y, e^{-\omega x} \sin \omega y$

$$u(0, y) = u(1, y) = 0, \quad y \in [0, 1]$$

$$u(x, y) = v(x) w(y)$$



$$u(0, y) = v(0) = v(1) = 0$$

Analytical solution of the 2D Laplace equation

$$v(x) = \begin{cases} A \cos \omega x + B \sin \omega x & \lambda < 0 \\ C_1 x + C_2 b & \lambda = 0 \\ C_3 e^{-\omega x} + C_4 e^{-\omega x} & \lambda > 0 \end{cases}$$

BC: $v(0) = 0$ \rightarrow $v(0) = \begin{cases} A \cos \omega x + 0 & \rightarrow A = 0 \\ 0 + C_2 & = 0 \rightarrow C_2 = 0 \\ C_3 e^{-\omega \cdot 0} + C_4 e^{-\omega \cdot 0} & \rightarrow -C_3 = C_4 := \frac{C}{2} \end{cases}$

$$v(x) = \begin{cases} B \sin \omega x & \lambda < 0 \\ C_1 x & \lambda = 0 \\ C \frac{e^{\omega x} - e^{-\omega x}}{2} = C \sinh x & \lambda > 0 \end{cases}$$

BC: $v(1) = 0$ \rightarrow $v(1) = \begin{cases} B \sin \omega & \rightarrow \omega_k = k\pi, k = 1, 2, 3, \dots \\ C_1 & = 0 \rightarrow C_1 = 0 \\ C \sinh(1) & \rightarrow C = 0 \end{cases}$ leads to $v(x) = 0$ (trivial solution)

$\neq 0$

Analytical solution of the 2D Laplace equation

$$v'' - \lambda v = 0,$$

$$w'' + \lambda w = 0$$

λ	$v(x)$	$w(y)$	$u(x, y) = v(x) w(y)$
$\lambda = -\omega^2 < 0$	$\cos \omega x, \sin \omega x$	$e^{-\omega y}, e^{\omega y},$	$e^{\omega y} \cos \omega x, e^{\omega y} \sin \omega x,$ $e^{-\omega y} \cos \omega x, e^{-\omega y} \sin \omega x$
$\lambda = 0$	$1, x$	$1, y$	$1, x, y, xy$
$\lambda = \omega^2 > 0$	$e^{-\omega x}, e^{\omega x}$	$\cos \omega y, \sin \omega y$	$e^{\omega x} \cos \omega y, e^{\omega x} \sin \omega y,$ $e^{-\omega x} \cos \omega y, e^{-\omega x} \sin \omega y$



$$v_k(x) = B_k \sin(\omega_k x) \quad \omega_k = k\pi, k = 1, 2, 3, \dots$$

Analytical solution of the 2D Laplace equation

$$v'' - \lambda v = 0,$$

$$w'' + \lambda w = 0$$

λ	$v(x)$	$w(y)$	$u(x, y) = v(x) w(y)$
$\lambda = -\omega^2 < 0$	$\cos \omega x, \sin \omega x$	$e^{-\omega y}, e^{\omega y},$	$e^{\omega y} \cos \omega x, e^{\omega y} \sin \omega x,$ $e^{-\omega y} \cos \omega x, e^{-\omega y} \sin \omega x$

$$u(0, y) = u(1, y) = 0, \quad y \in [0, 1] \quad \checkmark$$

$$u(x, 0) = 0, \quad x \in [0, 1]$$

$$\longrightarrow w(0) = 0 \quad w_k(0) = C_5 e^{-\omega_k \cdot 0} + C_6 e^{-\omega_k \cdot 0}$$

$$w_k(0) = C_{k5} + C_{k6} = 0 \quad \longrightarrow \quad -C_{k5} = C_{k6} := \frac{D_k}{2}$$

$$w_k(y) = D_k \frac{e^{\omega_k y} - e^{-\omega_k y}}{2} = D_k \sinh(\omega_k y)$$

$$\underbrace{\hspace{10em}}_{\sinh(\omega_k y)}$$

Analytical solution of the 2D Laplace equation

$$u_k(x, y) = v_k(x)w_k(y) = \sin(\omega_k x) \sinh(\omega_k y)$$

$$u(x, y) = \sum_{k=1}^{\infty} a_k v_k(x) w_k(y) = \sum_{k=1}^{\infty} a_k \sin(\omega_k x) \sinh(\omega_k y)$$


Analytical solution of the 2D Laplace equation

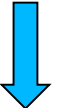
$$u(0, y) = u(1, y) = 0, \quad y \in [0, 1] \quad \checkmark$$
$$u(x, 0) = 0, \quad x \in [0, 1] \quad \checkmark$$

$$u(x, 1) = -x^2 + x, \quad x \in (0, 1)$$

$$u(x, y) = \sum_{k=1}^{\infty} a_k \sin(\omega_k x) \sinh(\omega_k y) \quad \omega_k = k \pi$$

$$u(x, 1) = \sum_{k=1}^{\infty} a_k \sin(\omega_k x) \sinh(\omega_k) = -x^2 + x$$

write the Fourier series of the initial condition : $-x^2 + x = \sum_{k=1}^{\infty} g_k \sin(\omega_k x)$  g_k



$$a_k = \frac{g_k}{\sinh(\omega_k)}$$