$\leftarrow$ Stab. of. Dyn. Sys.

Lyapunov

Let x^* be eq. pt of ⊙

x^* is stable if $\forall \varepsilon \, \exists \delta$

such that: $\|x_0 - x^*\| \leq \delta$ ⊙

Criteria (general):

ODE

- $\dot{x} = f(x)$, d.h. $f(x^*) = 0$
- $\|x(t) - x^*\| < \varepsilon \quad \forall t$
- $x(t) = x_0 + \int_{t_0}^{t} f(x)\,dt$

$$\approx x_0 + \int_{t_0}^{t} \{ \underbrace{f(x^*)}_{0} + D_x f (x - x^*) \}\,dt$$

$\rho(D_x f) \leq 0$ (≤ 0) 3.23

Lin case: $\dot{x}(t) = x_0 + \int A x \,dt$

for some ODE . . .

Diff. eq.

- $\alpha(x_n) = 0$, d.h. $x_{n+1} = x_n$ for $x_0 = x^*$
- $\|x_n - x^*\| < \varepsilon \quad \forall n$

Lin: $x_{n+1} = B x_n$, $\rho(B) < 1$ (≤ 1 under add. restriction (fully diagonalisable) 3.33 Deuflhard)

30 min

for some h $\iff$ stability of the induced difference scheme

Stable scheme . . .

$\dot{x} = 0$ (zero stability)

- very theoretical!
- necessary for convergence

inherit to One-Step Method:

$x_{n+1} = x_n + h \cdot \mathbb{b} \cdot \vec{0} = x_n$ obviously zero stable.

to Multistep Method: $\sum_{j=k}^{0} \alpha_j x_{n+j} = h \cdot \sum \beta_j \cdot 0 = 0$

as system:

$$\begin{pmatrix} x_n \\ \vdots \\ x_{n+k} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 & & \\ & & \ddots & \\ & & 0 & 1 \\ \alpha_0 & \dots & & \alpha_{k-1} \end{pmatrix}}_{\rho(\;\cdot\;) \leq 1} \begin{pmatrix} x_{n-1} \\ \vdots \\ x_{n+k-1} \end{pmatrix}$$

Remarks: same for expl/impl

- Seen also for Dahlquist for $\lambda h = 0 \Rightarrow$ „zero" stability
- $\Rightarrow$ does not depend on h.

40 min

- upper bound of

$\dot{x} = \lambda x$. „Dahlquists Problem". Not only $\lambda \sim 0$ important:

Remember HW: $\dot{x} = -1000x$, $h = 2000$. $h > 2000$: instable!

inherit to 1-Step method.

expl: $k_1 = \lambda x_n$

$k_2 = \lambda(x_n + h \cdot a_{21} \lambda x_n) = \lambda(Id + h a_{21} \lambda) x_n$

$k_3 = \lambda(x_n + h a_{32} \lambda(x_n + h a_{21} \lambda x_n) + h a_{32} \lambda x_n) = \lambda \cdot P(\lambda h) x_n$

$\vdots$

$\Rightarrow x_{n+1} = x_n + h \cdot \mathbb{b} \cdot k = Pol(h\lambda) x_n$

– never stable $\forall h$

impl:

$\mathbb{k} = \lambda(x_n \cdot \mathbb{1} + h \cdot A \mathbb{k})$

$\iff \mathbb{k} = \lambda(Id - h\lambda A)^{-1} \mathbb{1} \cdot x_n$

$\Rightarrow x_{n+1} = Id + h \cdot \lambda \cdot \mathbb{b}(Id - h\lambda A)^{-1} \mathbb{1} x_n$

$\exists A$: such that stable $\forall h$. (A-stable)

60 min

inherit to lin. Multistep Methods:

expl: $x_{n+k} + \alpha_{k-1} x_{n+k-1} + \dots = h\lambda(\beta_{k-1} x_{n+k-1} + \dots)$

$\iff x_{n+k} = (h\lambda\beta_{k-1} - \alpha_{k-1}) x_{n+k-1} + (h\lambda\beta_{..}) x_{n+k+k}$

never stable $\forall h$.

impl.:

$\dots = h\lambda(\beta_{+k} x_{n+k} + \dots)$

$\iff x_{n+k} = (1 - \lambda h \beta_k)^{-1} \cdot h\lambda(\beta_k \dots)$

– can be stable $\forall h$.

In fact (non-obvious): less stable Methods than 1-Step.

Q: Why not h small for a quick problem?

A: Yes, but $\dot{x} = \lambda x$ stands for $\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 \cdot x \\ \lambda x \end{pmatrix}$ slow / quick,

x_2 vanishes, has in fact no influence, but enforces small h.

„Stiffness"!

Def. A-stab: scheme is stable for all λh: $Re(\lambda h) < 0$.

Conclusions:
- A-stability is important in most cases. you often do not see the effect of instability – is it physics or error?
- explicit methods never have it, but are quick