Introduction to Scientific Computing

(Lecture 8: Ordinary differential equations)

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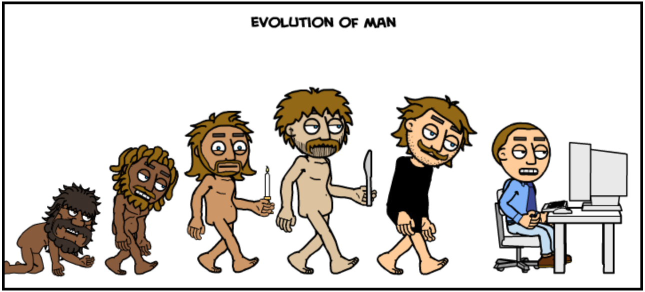
Modelling of reality by ODEs

ODEs are used to model **time dependent** *or so-called* **dynamical** *systems.* These systems are described by a **state** (vector of quantities describing the system) and **evolution rule**. The evolution rule is a fixed law which describes the future states of the system.



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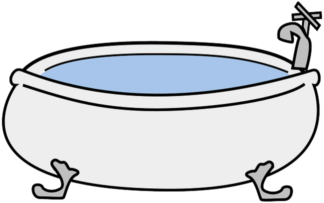
Example: Bathtub

State is the water level given in

- inital time: h_0 (known)
- arbitrary time t: $h(t)$ (not known)

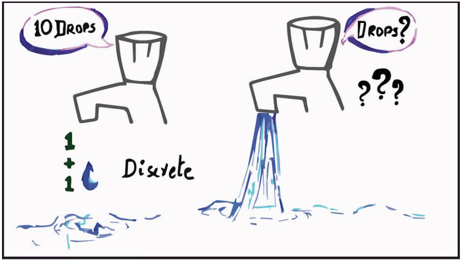


From **fluid mechanics** (*your expert knowledge*) we know that the speed of running water $\frac{dV(t)}{dt} = \frac{dAh(t)}{dt}$ is proportional to the depth of bathtub $h(t)$:

$$\frac{dV(t)}{dt} = A\frac{dh(t)}{dt} = -kh(t)$$

$$\Rightarrow \frac{dh(t)}{dt} = -\frac{k}{A}h(t) \quad \textbf{evolution law}$$

Classification of a dynamical system



Discrete vs. Continuous

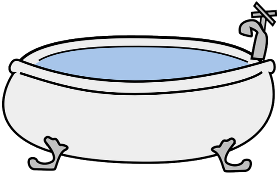
Difference equations vs Differential equations

The discrete system is represented by a **finitely many** system states. Typical example of such an system is **the bank account**.

$$\Delta x_n = \beta x_n$$

Continuous dynamical system

The system is characterised by an **infinitely many** system states.

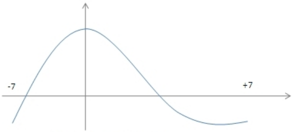


The evolution law

$$\frac{dh}{dt} = -\frac{k}{S}h$$

represents the **differential equation**.

Continuous to discrete



Continuous Signal
(takes values *in the set* [-7,7])

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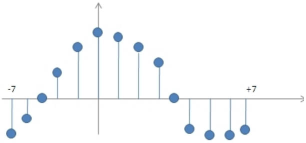


Discrete Signal
(takes values *at the integers* {-7,-6...0...6,7})

By **computer** simulation we transform

continuous system to discrete system,

or better to say

differential equation to difference equation.

Ordinary differential equations

The ordinary differential equation can be written as

$$F\left(t, x, x', x'',\ \cdots,\ x^{(n)}\right) = 0$$

in which F is in general nonlinear function and $x^{(n)}$ is the n-th derivative of dependent variable x with respect to independent time variable t. The initial conditions are given by $x(0) = x_0$. Thus, the ODE is also called **the initial value problem**.

Explicit vs implicit

The ordinary differential equation can be further classified to

- explicit

$$G\left(t, x, x', \cdots x^{(n-1)}\right) = x^{(n)}$$

- or implicit

$$F\left(t, x, x', x'', \ \cdots, \ x^{(n)}\right) = 0$$

equations, and

- autonomous (F does not depend on t explicitely)
- or non-autonomous (otherwise).

Exercise

- Explicit:

$$\frac{dx}{dt} = 5x$$

- Implicit

$$x(\frac{dx}{dt})^2 + t = 1$$

Exercise

- Autonomous:

$$\frac{dx}{dt} = 5x$$

- Non-autonomous

$$\frac{dx}{dt} = 5x + t$$

$$x\frac{dx}{dt} + t = 1$$

Non-autonomous ODEs

Every non–autonomous system can be converted into an autonomous one by adding a state variable $\mathbf{x}_{d+1} := t$. Hence, the non-autonomous system

$$\dot{\mathbf{x}}(t) = \frac{d}{dt}\mathbf{x}(t) = \mathbf{f}(t, \mathbf{x}(t)),\ \mathbf{x}(0) = \mathbf{x}_0,$$

is equivalent to the following autonomous one:

$$\begin{pmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_d \\ \dot{x}_{d+1} \end{pmatrix} = \begin{pmatrix} \mathbf{f}\left(x_{d+1}, (x_1, \ldots, x_d)\right) \\ 1 \end{pmatrix}.$$

Exercise

Let us transform non-autonomous

$$\frac{dx}{dt} = f(x,t) = 5x + t$$

into autonomous by taking $x_1 = x$ and $x_2 = t$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} f(x,t) \\ 1 \end{pmatrix} = \begin{pmatrix} 5x + t \\ 1 \end{pmatrix} = \begin{pmatrix} 5x_1 + x_2 \\ 1 \end{pmatrix}$$

Order of ODE

The ordinary differential equation can be classified according to the order to

- the first order ODE

$$F(t, x, x') = 0$$

- and higher order equatons

$$F\left(t, x, x', x'',\ \cdots,\ x^{(n)}\right) = 0$$

The order of ODE represents the highest order of derivative in equation.

Exercise

- First order

$$\frac{dx}{dt} = 5x$$

- Second order

$$\frac{d^2x}{dt^2} = 5x + t$$

- Third order

$$\frac{d^3x}{dt^3} = -x$$

Higher order ODE

Higher order ODE can be transformed to a first order ODE by taking

$$y_1 = x, y_2 = \dot{x}, y_3 = \ddot{x}, \ldots, y_k = x^{(k-1)},$$

and obtaining an equivalent representation of the ODE as

$$\dot{\mathbf{y}} = \begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \vdots \\ \dot{y}_k \end{pmatrix} = \begin{pmatrix} y_2 \\ y_3 \\ \vdots \\ y_k \\ F(t, y_k, \ldots, y_1) \end{pmatrix}.$$

Exercise

Let us transform

$$\frac{d^3x}{dt^3} = f(x) = -x$$

into first order by taking

$$y_1 = x, \quad y_2 = \dot{x}, \quad y_3 = \ddot{x}, \quad f(x) = -y_1$$

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \end{pmatrix} = \begin{pmatrix} y_2 \\ y_3 \\ f(x) \end{pmatrix} = \begin{pmatrix} y_2 \\ y_3 \\ -y_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

Conclusion

Considering first order autonomous ODE is enough.

Relevant Example: LInear first order ODE

The linear ordinary differential equation has a form

$$\frac{dx}{dt} = ax, \quad x(t_0) = x_0$$

to which correspond the solution

$$\frac{1}{x}dx = adt \Rightarrow x = x_0 e^{at}$$

and the equilibrium point

$$0 = ax_* \Rightarrow x_* = 0,$$

respectively.

Linear Systems of ODEs

Linear systems of ODEs with constant coefficients are only slightly more complicated than one single equation. They can be written as

$$\dot{\mathbf{x}} = A\mathbf{x}, \qquad \mathbf{x}(0) = \mathbf{x}_0,$$

where $A \in \mathbb{R}^{d\times d}$ is a constant matrix. It is again obvious that $\mathbf{x}_* = 0$ is an equilibrium point. As before, we try to express the solution in the form of the exponential ansatz

$$\mathbf{x}(t) = \mathbf{v}e^{\alpha t},$$

where $\mathbf{v}$ is a fixed vector.

Linear Systems of ODEs

Inserting this ansatz into the ODE, we obtain

$$\mathbf{v}\,\alpha\,e^{\alpha t} = A\left(\mathbf{v}e^{\alpha t}\right) = e^{\alpha t}A\mathbf{v}.$$

As $e^{\alpha t} \neq 0$, this becomes

$$A\mathbf{v} = \alpha\mathbf{v},$$

i.e. the ansatz is a solution if $\mathbf{v}$ is an eigenvector and α the corresponding eigenvalue of the matrix A.

Linear Systems of ODEs

For simplicity, we shall now assume that the matrix A is diagonisable (A has a full set of linearly independent eigenvectors— the eigenvectors form a basis in $\mathbb{R}^d$). Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_d\}$ be these eigenvectors and $\{\lambda_1, \ldots, \lambda_d\}$ be the corresponding eigenvalues (which may be complex), i.e. we have

$$A\mathbf{v}_j = \lambda_j \mathbf{v}_j,$$

and each function of the form

$$\mathbf{y}(t) = \mathbf{v}_j e^{\lambda_j t}$$

satisfies the ODE, as well as any linear combination.

Linear Systems of ODEs

Since $\{\mathbf{v}_1, \dots, \mathbf{v}_d\}$ is basis, we may find coefficients $\{\gamma_1, \dots, \gamma_d\}$, such that

$$\mathbf{x}_0 = \sum_{j=1}^{d} \gamma_j \mathbf{v}_j$$

holds. Then

$$\mathbf{x}(t) = \sum_{j=1}^{d} \gamma_j e^{\lambda_j t} \mathbf{v}_j$$

satisfies both the ODE and the initial condition, and hence is a solution.

Linear Systems of ODEs - the Ansatz

The Ansatz can be seen as justified by expanding the solution as

$$\mathbf{x}(t) = \sum_{j=1}^{d} \beta_j(t)\mathbf{v}_j.$$

As the solution varies in time, so do the coefficients $\beta_j(t), j = 1 \ldots d$. Inserting this into the ODE

$$\dot{\mathbf{x}} = A\mathbf{x}, \qquad \mathbf{x}(0) = \mathbf{x}_0,$$

we arrive at

$$\sum_{j=1}^{d} \dot{\beta}_j(t)\mathbf{v}_j = A\left(\sum_{j=1}^{d} \beta_j(t)\mathbf{v}_j\right) = \sum_{j=1}^{d} \beta_j(t)A\mathbf{v}_j = \sum_{j=1}^{d} \beta_j(t)\lambda_j\mathbf{v}_j,$$

Linear Systems

Or

$$\sum_{j=1}^{d}(\dot{\beta}_j(t) - \beta_j(t)\lambda_j)\mathbf{v}_j = 0.$$

As $\{\mathbf{v}_1, \dots, \mathbf{v}_d\}$ are linearly independent, the previous relation can only hold if

$$\dot{\beta}_j(t) = \lambda_j \beta_j(t),$$

which is a single linear ODE. Taking into account the initial condition, where $\beta_j(0) = \gamma_j$, we get the solution

$$\beta_j(t) = \gamma_j e^{\lambda_j t}.$$

Linear Systems of ODEs

However, note that λ_j can be complex

$$\lambda_j = p_j + i\omega_j$$

where $p_j = \mathrm{Re}(\lambda_j),\ \omega_j = \mathrm{Im}(\lambda_j)$ and i is the imaginary unit. In such a case the exponential becomes

$$e^{\lambda_j t} = e^{(p_j + i\omega_j)t} = e^{p_j t} e^{i\omega_j t}.$$

As $\left| e^{i\omega t} \right| = 1$ (a pure oscillation with frequency ω), stability or instability is determined by the factor $e^{p_j t}$.

FIRST ORDER ODE

First order ODE theory

Let be given the first order ODE

$$F(t, x, x') = 0, \quad x(0) = x_0$$

The following questions arise:

- under which conditions the previous equation has solution?
- if exists, is the solution unique?

A solution of this ODE on an interval $I \subset \mathbb{R}$ is a function $x : I \to \mathbb{X}^n$ for which x' exist at each $t \in I$, and

$$\forall t \in I \quad F(t, x(t), x'(t)) = 0$$

Note that $t_o \in I$ and $\mathbb{X}$ is either $\mathbb{R}$ or $\mathbb{C}$. This further implies that $x \in C(I)$ as well as $x' \in C(I)$ (space of continuous functions).

Solution of first order ODE

Let us observe the first order ODE

$$\frac{dx}{dt} = f(x,t), \quad x(t_0) = x_0$$

and let $f : I \to \mathbb{X}^n$ be a continuous function (meaning that for each $\hat{t} \in I \quad \lim_{t \to \hat{t}} f(t) = f(\hat{t})$). Then by integrating the previous equation from $t_0 \in I$

$$x(t) = x(t_0) + \int_{t_0}^{t} x'(s)ds = x(t_0) + \int_{t_0}^{t} f(x(s), s)ds.$$

Solution of first order ODE

So, $x(t)$ is the solution of the ODE

$$\frac{dx}{dt} = f(x,t), \quad x(t_0) = x_0$$

if and only if $x(t)$ is solution of the integral equation (IE)

$$x(t) = x(t_0) + \int_{t_0}^{t} x'(s)ds = x(t_0) + \int_{t_0}^{t} f(x(s),s)ds$$

-Allows us to study the ODEs via IEs. Has shape

$$x = F(x) \quad !$$

Solving IEs: Picards iteration

To compute the solution of IE one may use the fixed point iteration (also called Picard-Lindelöf iteration):

$$x^{(k+1)}(t) - x_0 = \int_{t_0}^{t} f(x^{(k)}(s), s)ds = F(x^{(k)}, s)$$

by starting from $x^{(0)} = x_0$.

Existance and uniqueness of the solution

Hence, the Banach fixed point theorem has to be satisfied:

- the mapping $F : \mathcal{X} \to \mathcal{X}$
- the mapping must be Lipschitz continuous
- the mapping must be contractive

which we seek to prove.

Completeness

Let us observe all continuous functions x which satisfy

$$x(t_0) = x_0, \quad |x - x_0| \leq r$$

when

$$t_0 < t < b = t_0 + h, \quad 0 < h \leq \alpha.$$

This is some "interval" $\mathcal{X} := \{x(t_0) = x_0, \quad |x - x_0| \leq r\}$. To prove Picard's theorem, we have to prove that

$$x = x_0 + \int_{t_0}^{t} f(x,s)ds \Rightarrow x - x_0 = \int_{t_0}^{t} f(x,s)ds$$

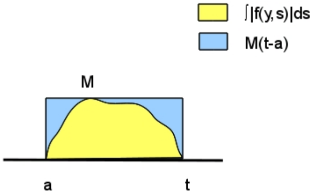
is a mapping from $\mathcal{X}$ to $\mathcal{X}$.

Completeness

Taking absolute values one obtains inequality

$$|x - x_0| \leq \int_{t_0}^{t} |f(x, s)| ds$$

Denote the maximum of $f(x, t)$ on the rectangle R as M, then the integral as area under function is smaller than the area of rectangle (width=h, height=M), i.e.



$$|x - x_0| \leq M(t - t_0)$$

As $t_0 < t < t_0 + h$, then

$$|x - x_0| \leq M(t - t_0) \leq M(t_0 + h - t_0) = Mh$$

Completeness

We started from

$$|x - x_0| \leq r$$

Hence, to have self-mapping the right hand side of inequality

$$|x - x_0| \leq Mh$$

has to be smaller than r, i.e.

$$|x - x_0| \leq Mh \leq r$$

If this is satisfied (h smaller than r/M) then F is a mapping from $\mathcal{X} \rightarrow \mathcal{X}$. Now we need to prove that the mapping is contraction.

Contraction

To prove contraction one has to assume that f is Lipschitz continuous w.r.t. to x:

Definition

Let $I \subset \mathbb{R}$ be an interval and $\mathcal{X} \subset \mathbb{X}^n$. We say that $f(t,x)$ mapping $I \times \mathcal{X}$ into $\mathbb{X}^n$ is uniformly Lipschitz continuous with respect to x if there is a constant L (called the Lipschitz constant) for which $\forall t \in I, \quad \forall x, y \in \mathcal{X}$

$$|f(t,x) - f(t,y)| \leq L|x - y|$$

We say that f is in (C, Lip) on $I \times \mathcal{X}$ if f is continuous on $I \times \mathcal{X}$ and f is uniformly Lipschitz continuous with respect to x on $I \times \mathcal{X}$.

Then the integrals

$$y = x_0 + \int_{t_0}^{t} f(y,s)ds, \quad x = x_0 + \int_{t_0}^{t} f(x,s)ds$$

give

$$y - x = \int_{t_0}^{t} (f(y,s) - f(x,s))ds$$

Contraction

By taking absolute values:

$$|y - x| \leq \int_{t_0}^{t} |f(y, s) - f(x, s)| ds \leq \int_{t_0}^{t} L|y - x| ds$$

Furthermore

$$\int_{t_0}^{t} L|y - x| ds \leq \int_{t_0}^{t} L \max|y - x| ds = L(t - t_0) \max|y - x|$$

Hence

$$|y - x| \leq L(t - t_0)d = Lhd$$

where $d := \max|y - x|$. This means that x is a continuous mapping and contractive when $Lh < 1$.

Existence and uniqueness

Theorem

Let $I = [t_0, t_0 + \beta]$ *and* $\mathcal{X} = \overline{Br(x_0)} = x \in \mathbb{X}^n : |x - x_0| \leq r$*, and suppose* $f(t,x)$ *is in (C, Lip) on* $I \times \mathcal{X}$*. Then there exisits* $\alpha \in (0, \beta]$ *for which there is a unique solution of the integral equation*

$$x(t) = x(t_0) + \int_{t_0}^{t} x'(s)ds = x(t_0) + \int_{t_0}^{t} f(x(s), s)ds$$

in $C(I_\alpha)$*, where* $I_\alpha = [t_0, t_0 + \alpha]$*. Moreover, we can choose* α *to be any positive number satisfying* $\alpha \leq \beta, \alpha \leq \frac{r}{M}$ *and* $\alpha < \frac{1}{L}$*, where* $M = \max_{(t,x) \in I \times \mathcal{X}} |f(t,x)|$ *and* L *is the Lipschitz constant for* f *in* $I \times \mathcal{X}$*.*

Picards iteration

Theorem (Global existance)

Let $I = [t_0, t_0 + \beta]$, *and suppose* $f(t, x)$ *is in (C, Lip) on* $I \times \mathbb{X}^n$. *Then there exists a solution* $x(t)$ *of the integral equation (IE) in* $C(I)$.

Theorem (Local existance)

Let $I = [t_0, t_0 + \beta]$ *and* $\mathcal{X} = \overline{Br(x_0)} = x \in \mathbb{X}^n : |x - x_0| \leq r, $, *and suppose* $f(t, x)$ *is in (C, Lip) on* $I \times \mathcal{X}$. *Then there exists a solution* $x(t)$ *of the integral equation (IE) in* $C(I_\alpha)$, *where* $I_\alpha = [t_0, t_0 + \alpha]$, $\alpha = \min(\beta, \frac{r}{M})$, *and* $M = \max(t, x) \in I \times \mathcal{X} |f(t, x)|$.

Corollary

$|x - x_0| \leq \frac{M_0}{L}(e^{L(t-t_0)} - 1)$ *for* $t \in I$ *or* $t \in I_\alpha$ *where* $M_0 = \max_{t \in I} |f(t, x_0)|$ *or* $M_0 = \max_{t \in I_\alpha} |f(t, x_0)|$

Stability of first order ODE

The dynamical system is in equilibrium state when the change of its state in time is equal to zero:

$$\frac{dx}{dt} = f(x) = 0$$

This further means that the equilibrium state x_* satisfies the nonlinear (or linear) equation

$$f(x_*) = 0$$

To find the root of the previous equation, one may use any of the previously studied methods such as Newton-Raphson procedure, etc.

Stability of first order ODE

The Lyapunov stability of equilibrium point x_* is classified as for difference equations, only that index n is repalced by time t.

$$x_0 = x_* + \delta$$

where δ is small perturbation. Hence, the definition of the stability is similar as in case of the difference equations.

Stability of first order ODE

Let x_* be an equilibrium of the ODE $\dot{x}(t) = f(t, x(t)),\ x(0) = x_0$.

1. x_* is called stable if

$$\forall \epsilon > 0\ \exists \delta > 0\ \forall x_0 :\ \|x_0 - x_*\| \le \delta \Longrightarrow \forall t > 0 :\ \|x(t) - x_*\| \le \epsilon.$$

2. x_* is an attractor or attractive if there is an $\delta > 0$ such that

$$\forall x_0 :\ \|x_0 - x_*\| \le \delta \Longrightarrow \|x(t) - x_*\| \longrightarrow 0 \ \text{ as } t \longrightarrow \infty.$$

3. x_* is asymptotically stable if x_* is stable and an attractor.
4. x_* is unstable if x_* is not stable.
5. x_* is called exponentially stable if x_* is asymptotically stable and there is an $a > 0$ and $C > 0$ such that

$$\forall t \ge 0 :\ \|x(t) - x_*\| \le Ce^{-at}.$$

Stability for linear first order ODE

The equilibrium point 0 of $\dot{x} = ax$ is

- stable and attractive (even exponentially stable) if $a < 0$. Why? Because $x = x_0 e^{at} \to 0$ when $a < 0$.

$$|\tilde{x}(t) - x_*| = |\tilde{x}| = |\tilde{x}_0| e^{at} \leq Ce^{-pt}, \quad p > 0$$

- stable if $a = 0$. Why? Because $x = x_0 e^{at} = x_0$.
- unstable if $a > 0$. Why? Because $x = x_0 e^{at} \to \infty$ when $a > 0$.

Linear systems of ODEs

Another way of solving first order systems that resembles the one-dimensional case closely is to write the solution of a linear first order system

$$\dot{\mathbf{x}} = A\mathbf{x}, \qquad \mathbf{x}(0) = \mathbf{x}_0,$$

abstractly as

$$\tilde{\mathbf{x}}(t) = e^{tA}\mathbf{x}_0,$$

where the matrix exponential function is used. This is defined just like the usual exponential function by its power series

$$e^t = \sum_{k=0}^{\infty} \frac{1}{k!} t^k \quad \Longrightarrow \quad \exp(tA) = e^{tA} = \sum_{k=0}^{\infty} \frac{1}{k!} t^k A^k.$$

Linear systems of ODEs

Of course, this expression cannot be utilised for practical computations. To obtain an expression which can be evaluated, assume that A is diagonisable and thus can be represented as $A = Q\Lambda Q^{-1}$, where Q contains the eigenvectors of A and Λ is a diagonal matrix consisting of the eigenvalues $\lambda_i, i = 1, \ldots, n$. This gives

$$e^{tA} = Qe^{t\Lambda}Q^{-1} = Q\,\mathrm{diag}(e^{t\lambda_i})\,Q^{-1}.$$

And so, in order to compute the explicit solution via the matrix exponential, again the eigenvectors and eigenvalues of A have to be found.

Exercise

Solve

$$y_1' = y_1$$
$$y_2' = y_1 - y_2$$

with i.c. $y_1(0) = 1, y_2(0) = 2$. The system reads

$$\dot{\mathbf{y}} = A\mathbf{y}$$

in which

$$A = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$$

The eigenvalues and eigenvectors are

$$\Lambda = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, V = \begin{pmatrix} 0 & 0.8944 \\ 1.0000 & 0.4472 \end{pmatrix}$$

Exercise

The solution can be written as

$$\mathbf{y} = c_1 e^{-t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + c_2 e^{t} \begin{pmatrix} 0.8944 \\ 0.4472 \end{pmatrix}$$

Constants can be found from initial conditions

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} = c_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 0.8944 \\ 0.4472 \end{pmatrix}$$

Equilibrium of linear Systems of ODEs

The equilibrium point of

$$\dot{\mathbf{x}} = A\mathbf{x}, \qquad \mathbf{x}(0) = \mathbf{x}_0,$$

is trivial since

$$\mathbf{x}_* = \mathbf{0}.$$

Stability of linear Systems of ODEs

To study the stability of linear systems of ODEs, one has to take the initial condition $\tilde{x}_0$ which is perturbed equilibrium point and to study the behaviour of the solution $\tilde{x}(t)$ in time. Hence, the same rules apply as on slide 25. However, note that the stability will now depend on the vector of eigenvalues $\lambda_j = p_j + i\omega_j$ in the following manner:

- If for any λ_j we have $\mathrm{Re}(\lambda_j) = p_j > 0$, then $x_* = 0$ is an unstable equilibrium.
- If for all λ_j we have $\mathrm{Re}(\lambda_j) = p_j \leq 0$, and at least one $p_j = 0$, then $x_* = 0$ is stable but not attracting/asymptotically stable.
- If for all λ_j we have $\mathrm{Re}(\lambda_j) = p_j < 0$, then $x_* = 0$ is asymptotically stable and even exponentially stable.

Exercise

For the ODE Solve

$$y_1' = y_1$$

$$y_2' = y_1 - y_2$$

the eigenvalues are $\lambda_1 = -1 < 0$ and $\lambda_2 = 1 > 0$. Hence, the system is not stable at equilibrium point $\mathbf{0}$.

Nonlinear systems of ODEs

In case of the nonlinear system of ODEs

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}, t), \quad \mathbf{x}(t_0) = \mathbf{x}_0$$

one may investigate stability by observing perturbation

$$\mathbf{y}(t) = \mathbf{x}(t) - \mathbf{x}_*(t)$$

where $\mathbf{x}$ is the solution for the perturbed initial point $\tilde{\mathbf{x}} = \mathbf{x}_{*,0} + \epsilon$.

Nonlinear systems of ODEs

The last equation

$$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y} + \mathbf{x}_*, t)$$

now can be expanded in Taylor series

$$\begin{aligned}
\dot{\mathbf{y}}(t) &= \mathbf{f}(\mathbf{x}_* + \mathbf{y}(t)) \\
&= \mathbf{f}(\mathbf{x}_*) + D\mathbf{f}(\mathbf{x}_*)\mathbf{y}(t) + O(\|\mathbf{y}(t)\|^2) \\
\dot{\mathbf{y}}(t) &\approx D\mathbf{f}(\mathbf{x}_*)\mathbf{y}(t),
\end{aligned}$$

such that one obtains linearised version of the system model around the equilibrium point.

Lyapunov stability

Theorem

Under certain conditions the results obtained in the stability analysis of linear systems can be applied to nonlinear systems:
Let $\mathbf{x}_*$ *be an equilibrium state of the ODE* $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$*. Let* $D\mathbf{f}(\mathbf{x}_*)$ *be the Jacobian matrix of* $\mathbf{f}$ *in* $\mathbf{x}_*$*. Then the following statements hold:*

(i) *If all eigenvalues* μ *of* $D\mathbf{f}(\mathbf{x}_*)$ *have negative real part,* $Re(\mu) < 0$*, then* $\mathbf{x}_*$ *is stable for the nonlinear ODE.*

(ii) *If there is at least one eigenvalue* μ *of* $D\mathbf{f}(\mathbf{x}_*)$ *with positive real part,* $Re(\mu) > 0$*, then* $\mathbf{x}_*$ *is unstable for the nonlinear ODE.*

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(iii) *If Re*$(\mu) \leq 0$ *for every eigenvalue* μ*, and if for at least one eigenvalue Re*$(\mu) = 0$*, then the nonlinear part of* $\mathbf{f}$ *determines the stability of the equilibrium and our theorem is not applicable.*

(iv) *Our definition of stability is stability in Lyapunov's sense, that is we investigate the stability with respect to a perturbation of the initial conditions. A perturbation of the governing equation (structural stability) is not considered here.*

Exercise

The nonlinear ODE

$$y_1' = y_1^2 - 1$$

can be linearised around the equilibrium point $y_* = 1$ such that

$$y_1' = Df(y_*)y_1 = 2y_1$$

holds. Since $2 > 0$ one concludes that this equilibrium point is not stable.