Introduction to Scientific Computing

(Lecture 7: Equilibrium points and Stability)

Bojana Rosić, Thilo Moshagen

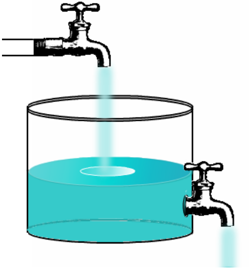
Institute of Scientific Computing

November 29, 2017

Equilibrium (stationary/fixed points)

The equilibrium of the dynamical system is a steady state which does not change in time:

$$\mathbf{x}_0 = \mathbf{x}_1 = \mathbf{x}_2 = \ldots = \mathbf{x}_n = \mathbf{x}_*.$$



Example:

The water level x_n in reservoir does not change as the amount of water that runs in is equal to the amount of water that comes out.

Equilibrium (stationary/fixed points)

Equilibrium 1

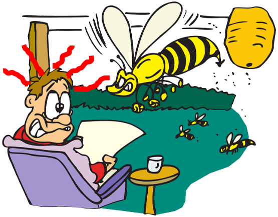
Equilibrium 2

Equilibrium 3

How to judge if equilibrium is stable or not?

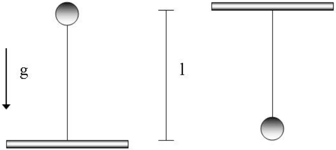
To see whether $\mathbf{x}_*$ is stable or not, use as initial condition $\mathbf{x}_0 = \mathbf{x}_* + \delta$ (**a little perturbation**) where $|\delta|$ is small. Then, evaluate $\mathbf{x}_n$ and see what is the behavior of the solution for $n \to \infty$? Does it converge towards some value, does it cycle periodically around some value, or does encounter some other behavior?

- if for $n \to \infty$ the solution $x_n \to \infty$, then the system is **unstable**
- if for $n \to \infty$ the solution $x_n \to x_*$ or near to it, then the system is **stable**



Example

Consider a pendulum. When the pendulum is pointing straight up then the system can have a steady state. However, minor changes will result in the pendulum swinging to either side. This steady state is an unstable steady state. Now consider the pendulum hanging down. When the pendulum is in is vertical position then the system does not change and this is also a steady state of a system. However, any perturbation of the pendulum will not have an effect in the long run as the pendulum will ultimately settle down to its steady state. This steady state is a stable steady state.



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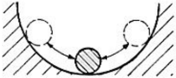
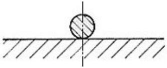
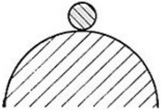
Stability

Assume that a system is in equilibrium. If the system is disturbed a little, one of the following may happen:

- The system might immediately return to the equilibrium. In this case, the equilibrium is called asymptotically stable.
- Or the system might move around in the neighborhood of the old equilibrium without returning to it. However, the system will not move far away. In this case, the equilibrium is stable but not asymptotically stable.
- The system might move far away from the old equilibrium. In this case, the equilibrium is instable.







Let us now transform this to math language

Stability of FODE

Consider: general difference equation of order 1 and dimension d

$$\mathbf{x}_{n+1} = F(\mathbf{x}_n), \qquad \mathbf{x}_n \in \mathbb{R}^d,\ n \in \mathbb{N}.$$

Definition: An equilibrium point of the dynamical system

$$\mathbf{x}_{n+1} = F(\mathbf{x}_n), \qquad \mathbf{x}_n \in \mathbb{R}^d,\ n \in \mathbb{N}$$

is a state–vector $\mathbf{x}_* \in \mathbb{R}^d$ such that

$$F(\mathbf{x}_*) = \mathbf{x}_*$$

holds.

Stability of FODE

Hence, the evaluation of equilibrium points can be formulated as Fixed Point search

$$F(\mathbf{x}_*) = \mathbf{x}_*$$

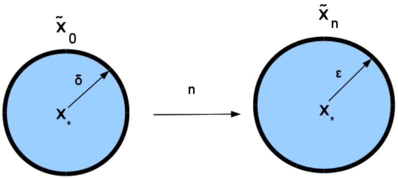
and vice versa, a Fixed Point of such F is an equilibrium point of the first order DE given by it.

Stability of FODE

Let $\mathbf{x}_* = F(\mathbf{x}_*) \in \mathbb{R}^d$ be an equilibrium point of the dynamical system $\mathbf{x}_{n+1} = F(\mathbf{x}_n)$ where $\mathbf{x}_n \in \mathbb{R}^d$, $n \in \mathbb{N}$. Furthermore, let

$$\tilde{\mathbf{x}}_{n+1} = F(\tilde{\mathbf{x}}_n), \qquad \tilde{\mathbf{x}}_n \in \mathbb{R}^d,\ n \in \mathbb{N}$$

be the **pertubed** solution.



Definition (1)

The equilibrium point $\mathbf{x}_*$ is called stable if for all $\epsilon > 0$ exist a $\delta > 0$ such that for all $\tilde{\mathbf{x}}_0$ coming from $\|\mathbf{x}_* - \tilde{\mathbf{x}}_0\| \leq \delta$ holds

$$\|\mathbf{x}_* - \tilde{\mathbf{x}}_n\| < \epsilon, \quad \forall n > 0.$$

Here, $\|\cdot\|$ stands for **the norm**.

Stability of FODE

Definition (2)

The equilibrium point $\mathbf{x}_*$ *is called* *attractive* *if there exists a* $\delta > 0$ *such that for all* $\tilde{\mathbf{x}}_0$ *coming from* $\|\mathbf{x}_* - \tilde{\mathbf{x}}_0\| < \delta$ *holds*

$$\lim_{n\to\infty} \|\mathbf{x}_* - \tilde{\mathbf{x}}_n\| = 0.$$

Definition (3)

The equilibrium point $\mathbf{x}_*$ *is called* *asymptotically stable* *if* $\mathbf{x}_*$ *is stable and attractive.*

Definition (4)

The equilibrium point $\mathbf{x}_*$ *is called* *unstable* *if it is not stable.*

Stability of First Order DE

For stability investigation we usually observe the normal and perturbed systems

System	Perturbed system
$\mathbf{x}_{n+1} = F(\mathbf{x}_n)$	$\tilde{\mathbf{x}}_{n+1} = F(\tilde{\mathbf{x}}_n)$
with equilibrium	with initial condition
$\mathbf{x}_* = F(\mathbf{x}_*)$	$\tilde{\mathbf{x}}_0 = \mathbf{x}_* + \delta$

which gives us the difference

$$\mathbf{y}_n := \tilde{\mathbf{x}}_n - \mathbf{x}_*$$

Hence, $\mathbf{x}_*$ is asymptotically stable if $\mathbf{y}_n \to 0$ for $n \to \infty$
and stable if $\mathbf{y}_n$ stays bounded.

Exercise: one equation

Let us observe

$$x_{n+1} = ax_n$$

where a is a scalar. Then the equilibrium point is

$$x_* = ax_* \Rightarrow x_* = 0.$$

The solution of the difference equation is something we have learned before and reads

$$x_n = a^n x_0.$$

To check if $x_* = 0$ is stable we need to perturb the initial condition

$$\tilde{x}_0 = x_* + \delta, \quad \delta > 0$$

Exercise: one equation

The new solution after perturbation reads

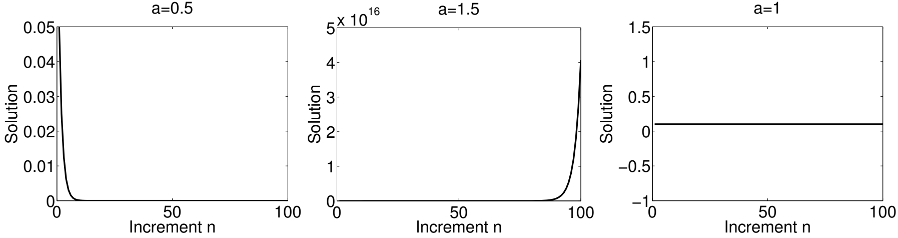
$$\tilde{x}_n = a^n(x_* + \delta).$$

Having that $x_* = 0$ the last equation becomes

$$\tilde{x}_n = a^n(\delta).$$

Therefore, whether $x_* = 0$ is stable (i.e. $\tilde{x}_n \to x_*$ when $n \to \infty$) or not depends on a.

Exercise: one equation



Exercise: one equation

- If $|a| > 1$ x_n will grow without bound. Therefore x_* is unstable.
- If $|a| < 1$, then x_n will converge to $x_* = 0$, i.e. for small perturbations δ, the system won't move away from the equilibrium but return to it. In this case, $x_* = 0$ is a stable equilibrium.
- if $|a| = 1$ then $x_n = \delta$, and hence the system moves around a neighborhood of the old equilibrium without returning to it. Hence, $x_* = 0$ is stable but not asymptotically

Exercise: system of equations

For a system of FODE

$$\mathbf{x}_{n+1} = A\mathbf{x}_n = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix} \mathbf{x}_n$$

we may find the equilibrium point

$$\mathbf{x}_* = A\mathbf{x}_* = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix} \mathbf{x}_* \Rightarrow \mathbf{x}_* = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

On the other hand, by diagonalization the solution of the previous system is

$$\mathbf{x}_n = A^n \mathbf{x}_0 = \sum_{j=1}^{3} c_j \lambda_j^n \mathbf{v}_j$$

Exercise: Stability of system of lin. DE

Stability criteria:

- If all λ_i of A have absolute value smaller than one: $(\forall i = 1, \ldots, d : |\lambda_i| < 1)$, then for every $\mathbf{x}_0 \in \mathbb{R}^d$ the sequence $\mathbf{x}_n \overset{n\to\infty}{\longrightarrow} 0$, and $\mathbf{x}_*$ is asymptotically stable.
- If any λ_i of A has absolute value greater than one: $(\exists i : |\lambda_i| > 1)$ then there exist $\mathbf{x}_0 \in \mathbb{R}^d$ such that the sequence $\mathbf{x}_n \overset{n\to\infty}{\longrightarrow} \infty$, and $\mathbf{x}_*$ is unstable.
- If all λ_i of A have absolute value smaller or equal than one: $(\forall i = 1, \ldots, d : |\lambda_i| \leq 1)$ and if there are λ_j's with $|\lambda_j| = 1$, then we cannot decide whether or not $\mathbf{x}_*$ is stable.

First two statements obvious, last stems from the Jordan (generalized Eigen-) decomposition of A. See e.g. Deuflhard/BornemannII, Th. 3.33

Stability of General First Order DE

Apply *Banachs FP Theorem*!!
Same thing, without mentioning Banach: If DF exists, using the Taylor expansion

$$\begin{aligned} \mathbf{y}_n &= \tilde{\mathbf{x}}_n - \mathbf{x}_* = F(\tilde{\mathbf{x}}_{n-1}) - \mathbf{x}_* \\ &= \underbrace{F(\mathbf{x}_*)}_{=\mathbf{x}_*} + DF(\mathbf{x}_*)(\tilde{\mathbf{x}}_{n-1} - \mathbf{x}_*) + \mathcal{O}(|\tilde{\mathbf{x}}_{n-1} - \mathbf{x}_*|^2) - \mathbf{x}_* \\ &= DF(\mathbf{x}_*)\mathbf{y}_{n-1} + \mathcal{O}(|\mathbf{y}_{n-1}|^2). \end{aligned}$$

Assume, that $|\mathbf{y}_{n-1}|$ is small. Then

$$\mathbf{y}_n = DF(\mathbf{x}_*)\mathbf{y}_{n-1},$$

in which $DF(\mathbf{x}_*)$ denotes the Jacobi–matrix of F in the equilibrium point $\mathbf{x}_*$.

Stability of First Order DE

The system

$$\mathbf{y}_n = DF(\mathbf{x}_*)\mathbf{y}_{n-1}.$$

is now linear FODE and we may diagonalise the matrix $F'(\mathbf{x}_*)$ such that it has d linearly independent eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_d \in \mathbb{R}^d$ and eigenvalues $\lambda_1, \ldots, \lambda_d \in \mathbb{C}$. Then there exist coefficients $c_1, \ldots, c_d$ such that

$$\mathbf{y}_0 = \sum_{j=1}^{d} c_j \mathbf{v}_j \quad \text{and} \quad \mathbf{y}_n = \sum_{j=1}^{d} c_j \lambda_j^n \mathbf{v}_j.$$

Now judging on λ values one may define the stability criteria.

Stability of First Order DE

Stability criteria:

- If all λ_i of $DF(\mathbf{x}_*)$ have absolute value smaller than one: $(\forall i = 1, \ldots, d : |\lambda_i| < 1)$, then for every $\mathbf{y}_0 \in \mathbb{R}^d$ the sequence $\mathbf{y}_n \overset{n\to\infty}{\longrightarrow} 0$, and $\mathbf{x}_*$ is asymptotically stable.
- If any λ_i of $DF(\mathbf{x}_*)$ has absolute value greater than one: $(\exists i : |\lambda_i| > 1)$ then there exist $\mathbf{y}_0 \in \mathbb{R}^d$ such that the sequence $\mathbf{y}_n \overset{n\to\infty}{\longrightarrow} \infty$, and $\mathbf{x}_*$ is unstable.
- If all λ_i of $DF(\mathbf{x}_*)$ have absolute value smaller or equal than one: $(\forall i = 1, \ldots, d : |\lambda_i| \leq 1)$ and if there are λ_j's with $|\lambda_j| = 1$, then higher order terms in the Taylor–series of F are required to decide whether or not $\mathbf{x}_*$ is stable.

See e.g. Deuflhard/BornemannII, Th. 3.33...
Note that this is nearly proving the Lipschitz condition, but one $\lambda = 1$ is allowed.

Exercise: system of equations

For a system of FODE

$$\mathbf{x}_{n+1} = A\mathbf{x}_n = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix} \mathbf{x}_n$$

we may find the equilibrium point

$$\mathbf{x}_* = A\mathbf{x}_* = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix} \mathbf{x}_* \Rightarrow \mathbf{x}_* = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

On the other side, we know that the solution of the previous system is

$$\mathbf{x}_n = A^n \mathbf{x}_0 = \sum_{j=1}^{3} c_j \lambda_j^n \mathbf{v}_j$$

Exercise: system of equations

To check if $\mathbf{x}_*$ is stable let us perturb the initial condition

$$\tilde{\mathbf{x}}_0 = \mathbf{x}_* + \delta, \quad \delta > 0$$

such that

$$\tilde{\mathbf{x}}_n = A^n \tilde{\mathbf{x}}_0 = A^n(\mathbf{x}_* + \delta)$$

holds. Having that $\mathbf{x}_* = \mathbf{0}$ one obtains

$$\tilde{\mathbf{x}}_n = A^n \delta = \sum_{j=1}^{3} m_j \lambda_j^n \mathbf{v}_j$$

Hence, $A^n \delta \to 0$ when $|\lambda_j^n| \to 0$.

Exercise: system of equations

In our case

$$\lambda_{1,2} = 0.3652 \pm 0.6916i, \quad \lambda_3 = 3.2695$$

which further means

$$|\lambda_{1,2}| = \sqrt{0.3652^2 + 0.6916^2} = 0.7821 < 1$$

and

$$|\lambda_3| = 3.2695 > 1$$

Thus, the equilibrium point 0 is **unstable**.
If there are no stable points, one calls the system unstable.