Introduction to Scientific Computing

(Lecture 4: Banachs Fixed-Point Theorem)

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Evaluating equilibrium point

To find the equilibrium of a dynamical system

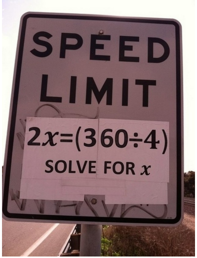
$$\mathbf{x}_* = F(\mathbf{x}_*),$$

a set of linear or nonlinear equations

$$G(\mathbf{x}_*) = 0,$$

has to be solved, where

$$G(\mathbf{x}) = F(\mathbf{x}) - \mathbf{x}.$$



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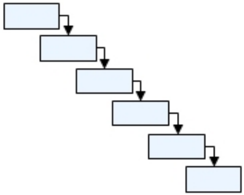
Solving set of equations: Fixed point iterations

Similarly, the system of equations

$$G(\mathbf{x}) = 0$$

can be solved by constructing a dynamical system with

$$F(\mathbf{x}_*) = \mathbf{x}_*$$



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in which $\mathbf{x}_*$ denotes the equilibrium point. Once the appropriate dynamical system is found, one may apply the evolution rule by starting from some initial point $\mathbf{x}_0$

$$\mathbf{x}_{n+1} = F(\mathbf{x}_n)$$

until $\mathbf{x}_{n+1}$ "**almost matches**" $\mathbf{x}_n$ (convergence).

Problem

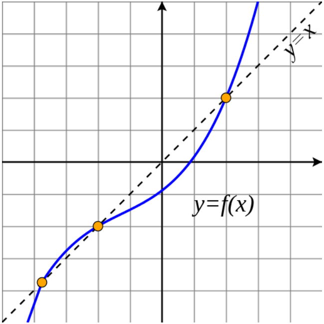
We want to solve

$$F(\boldsymbol{x}_*) = \boldsymbol{x}_*$$

such that

- the mapping F has the fixed point (solution)
- the fixed point $\mathbf{x}_*$ is unique
- can be obtained by iterative process

$$\mathbf{x}_{n+1} = F(\mathbf{x}_n)$$



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Exercise

Problem: Find the root 1.8556 *of the polynomial*

$$x^4 - x - 10 = 0$$

in an iterative manner.

First, one has to find the proper dynamical system $x_* = F(x_*)$. Let us try with

$$\text{I case} \quad F(x) = \frac{10}{x^3 - 1}$$

and the fixed point iterative scheme

$$x_{n+1} = \frac{10}{x_n^3 - 1}, \quad x_0 = 2$$

Exercise

Let us now try another one

$$\text{II case} \quad F(x) = (x + 10)^{1/4}$$

and the scheme

$$x_{n+1} = (x_n + 10)^{1/4}, \quad x_0 = 2.0$$

Exercise

Finally, let us try

$$\text{III case} \quad F(x) = \frac{(x+10)^{1/2}}{x}$$

and the scheme

$$x_{n+1} = \frac{(x_n+10)^{1/2}}{x_n}, \quad x_0 = 2.0$$

Comparison

For the same accuracy $abs(x_n - 1.8556) < 1e\text{-}3$ we get

Term	I case	II case	III case
No. of iterations	maxIter	4	55
Final value	1e-2	1.855	1.855

Hence, the first iterative procedure is not good. The other two are giving correct result. However, the second one is much faster and hence more suitable.

Thus, the question: *How to choose the proper scheme?*

How to choose

In order to choose the proper scheme we need to study convergence. This means that we need to study the error in the solution and its behaviour with respect to the number of iterations.

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The question is now how we will know this without doing calculations (apriori)? The answer is that the function

$$\mathbf{x} = F(\mathbf{x})$$

has to fulfill some conditions given by **Banach fixed point theorem**.

Problem

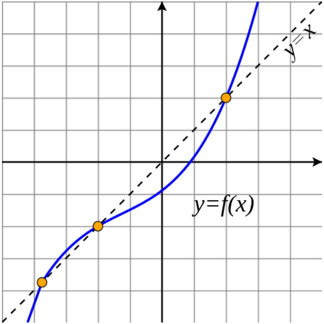
We want to solve

$$F(\boldsymbol{x}_*) = \boldsymbol{x}_*$$

such that

- the mapping F has the fixed point (solution)
- the fixed point $\mathbf{x}_*$ is unique
- can be obtained by iterative process

$$\mathbf{x}_{n+1} = F(\mathbf{x}_n)$$



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Fixed point iteration

The question is now how we will know this without doing calculations if the previous iteration will give us the result? The answer is that the function

$$\mathbf{x} = F(\mathbf{x})$$

has to fulfill some conditions given by **Banach fixed point theorem**.

Banach fixed point theorem

Definition

Let V be a Banach space, let $K \subset V$ be a closed subset, and let $F : K \longrightarrow K$ be a contraction ($0 \leq q < 1$) such that

$$\|F(\mathbf{x}) - F(\mathbf{y})\| \leq q\|\mathbf{x} - \mathbf{y}\| \qquad \text{for all } \mathbf{x}, \mathbf{y} \in K.$$

holds. Then the following conclusions hold:

1.) *F has a unique fixed point $\mathbf{x}_* \in K$, i. e. $F(\mathbf{x}_*) = \mathbf{x}_*$.*

2.) *For any $\mathbf{x}_0 \in K$ the sequence $\mathbf{x}_{n+1} = F(\mathbf{x}_n)$ converges to $\mathbf{x}_*$.*

Error estimates

Additionally, one may define

- A posteriori error estimate:

$$\|\mathbf{x}_* - \mathbf{x}_n\| \leq \frac{q}{1-q}\|\mathbf{x}_n - \mathbf{x}_{n-1}\|$$

- A priori error estimate:

$$\|\mathbf{x}_* - \mathbf{x}_n\| \leq \frac{q^n}{1-q}\|\mathbf{x}_1 - \mathbf{x}_0\|.$$

Lipschitz continuity (LC)

Definition

The function $F : K \to K$ *is called Lipschitz continuous if:*

$$\|F(x) - F(y)\| \leq q\|x - y\|$$

holds for any $x, y \in K$. *Here,* q *is some positive constant.*

Exercise

The function $F = x^2$ satsifies

$$\|x^2 - y^2\| = \|(x-y)(x+y)\| \leq \|(x+y)\|\|x-y\|$$

Hence, comapring to

$$\|x^2 - y^2\| \leq q\|x-y\|$$

one has that

$$q = \|(x+y)\|$$

Thus function is Lipschitz continuous only **locally** because for $x \to \infty \Rightarrow q \to \infty$.

LC of differentiable functions

Every differentiable function at x is Lipschitz continuous at x. Opposite does not hold.

LC of differentiable functions

Let us observe the equation of straight line connecting $\mathbf{y}$ and $\mathbf{x}$ in K

$$\xi = \mathbf{y} + \eta(\mathbf{x} - \mathbf{y}), \quad y \leq \xi \leq x$$

in which $\eta \in [0, 1]$. Then one may observe that

$$F(\xi) = F(\mathbf{y} + \eta(\mathbf{x} - \mathbf{y})) = G(\eta)$$

and

$$\frac{dG}{d\eta} = \frac{\partial F}{\partial \xi}\frac{\partial \xi}{\partial \eta}, \quad \frac{\partial \xi}{\partial \eta} = (\mathbf{x} - \mathbf{y})$$

Observing that

$$F(\mathbf{x}) = G(1), \quad F(\mathbf{y}) = G(0)$$

one may write

$$F(\mathbf{x}) - F(\mathbf{y}) = G(1) - G(0) = \int_0^1 \frac{dG}{d\eta} d\eta = \int_0^1 \frac{\partial F}{\partial \xi}(\mathbf{x} - \mathbf{y}) d\eta$$

LC of differentiable functions

Let F be differentiable, then the fundamental theorem of calculus asserts that there is an $\eta \in [0,1]$ such that

$$F(\mathbf{x}) = F(\mathbf{y}) + \int_0^1 F'(\xi)(\mathbf{x}-\mathbf{y})d\eta \qquad \text{for all } \mathbf{x}, \mathbf{y} \in \mathbb{R}^d,$$

$$\begin{aligned}
\|F(\mathbf{x}) - F(\mathbf{y})\| &\leq \int_0^1 \|F'(\xi)(\mathbf{x}-\mathbf{y})\| d\eta \\
&\leq \left(\int_0^1 \|F'(\xi))\| d\eta \right) \|\mathbf{x}-\mathbf{y}\| \\
&\leq \underbrace{\max_{\eta \in [0,1]} \|F'(\xi))\|}_{=:q} \|\mathbf{x}-\mathbf{y}\| \\
&= q\|\mathbf{x}-\mathbf{y}\|.
\end{aligned}$$

LC of differentiable functions

Accordingly, F has for a Lipschitz constant:

$$q := \sup_{\eta \in K} \|F'(\xi)\|$$

Exercise

For the function $F = x^2$ one has

$$\|F'(\xi)\| = 2\|\xi\|$$

i.e.

$$q := \sup_{\eta \in K} \|F'(\xi)\| = 2 \max \|\xi\|$$

Comapring to

$$q = \|(x + y)\|$$

one may conclude that we have obtained the same result havng in mind that for some $\xi \in K$

$$q = \|(x + y)\| \leq 2 \max \|\xi\|$$

LC of differentiable functions

In practical applications q is usually not known analytically. A lower bound can be computed while performing the iteration

$$\mathbf{x}_{k+1} = F(\mathbf{x}_k)$$

by setting

$$q_k := \frac{\|\mathbf{x}_{k+1} - \mathbf{x}_k\|}{\|\mathbf{x}_k - \mathbf{x}_{k-1}\|}, \qquad \hat{q} := \max_k q_k.$$

The value $\hat{q}$ obtained in this manner is often an acceptable estimate for q and it has the advantage that its computation is inexpensive.

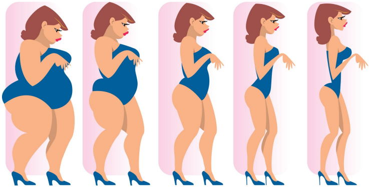
Contraction

Definition

A contraction mapping $F(x)$, or contraction or contractor is a function F from K to itself, with the property that there is some non-negative real number $0 \leq q < 1$ such that for all x and y in K

$$\|F(x) - F(y)\| \leq q\|x - y\|$$

holds.



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Exercise

For the function $F = x^2$ the Lipschitz constant

$$q := \sup_{\eta \in K} \|F'(\xi)\| = 2 \max \|\xi\|$$

is less than 1 only when $\|\xi\| < 0.5$. Otherwise, it is not. Hence, for $x \in [-0.5, 0.5]$ the function will be continuous and contractive.

Hence,

In order to solve

$$F(x) = x$$

we may apply fixed point iteration and obtain unique x_* from any $x_0 \in K$, only if $F(x)$ satisifes Banach fixed point theorem, i.e. if the mapping $F(x) : K \to K$ is Lipschitz continous and contractive.

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Exercise

In our example we had that the first scheme reads

$$x_{n+1} = \frac{10}{x_n^3 - 1}$$

The mapping $F(x) = \frac{10}{x_n^3 - 1}$ is defined on $\mathbb{R}$ and returns values in $\mathbb{R}$. However, we will show that $F(x)$ is not Lipschitz continuous on $\mathbb{R}$, but only on some intervals K. Now we need to find interval K such that $F(x) : K \to K$, i.e. for each $x \in K$ we get some $F(x) \in K$.

Exercise

The mapping

$$F(x) = \frac{10}{x_n^3 - 1}$$

is differentiable and has Jacobian

$$F'(x) = \frac{-30x^2}{(x_n^3 - 1)^2}$$

The derivative goes to infinity when x_n approaches 1. Hence, we need to observe interval K which does not contain 1. Let us observe the interval (a, ∞) where $a > 1$.

Exercise

The absolute value of Jacobian is

$$F'(x) = 30 \left| \frac{x^2}{(x^3 - 1)^2} \right|$$

and it represents the decrising function with increasing x. This means that the supremum happens at the beginning of interval a:

$$q = \sup |F'(x)| \geq F'(a)$$

For interval $(1, 2]$ this value is larger than 1 and hence the scheme is not contractive. This means that we cannot find the root 1.856 in this interval.

Exercise

On the other side, the second scheme

$$x_{n+1} = (x_n + 10)^{1/4}$$

defines the mapping characterised by a Jacobian

$$F'(x) = \frac{0.25}{(x_n + 10)^{3/4}}$$

The derivative goes to infinity when x_n approaches -10. Hence, we need to observe interval K which does not contain -10. Let us observe the interval (a, ∞) where $a > 0$ (in this interval lies 1.8556).

Exercise

The Jacobian is decreasing function with increasing x. Hence, the supremum becomes

$$q = \sup |F'(x)| \geq |F'(a)|$$

in which a is the beginning of interval. If we choose that the interval is $[1, 2]$. Then,

$$q = 4.6957e - 05$$

which tells us that the mapping is contractive. Thus, we may find the root 1.8566 using this scheme.

Exercise II

The difference equation

$$x_{n+1} - \frac{1}{6}(x_n^3 + x_{n-1}^2 + 1) = 0$$

can be rewritten to:

$$x_{n+1} = \frac{1}{6}(x_n^3 + x_{n-1}^2 + 1) =: F(x_n).$$

Thus, the fixed (equilibrium) point satisfies:

$$x_* = F(x_*)$$

and has values

$$x_{*1} = -3.0644, x_{*2} = 1.8920, x_{*3} = 0.1725$$

Exercise II

However, we would like to obtain them numerically by solving

$$x_*^{(k+1)} = F(x_*^{(k)}), \quad x_* \in K$$

The question is if we can do that? Yes, if

- $K = (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$ is complete
- F is locally or globally Lipschitz continuous on K
- and Lipschitz constant is $0 \leq q < 1$

Exercise II

Having

$$F(x) = \frac{1}{6}(x^3 + x^2 + 1)$$

let us check for $a = -1.2$ in $K = (-\infty, -1)$ the value

$$F(a) = F(-1.2) = 0.1187$$

Hence, $F(a) \notin K$ and we may conclude that this interval is not complete. Similarly by taking $a = 1.2$ in $K = (1, \infty)$ we may show that

$$F(a) = 0.6947 \notin K$$

Hence, the interval $K = (1, \infty)$ is also not complete.

Exercise II

If we take $K = (-1, 1)$ we may show that for each a in K the value $F(a)$ falls into K (if you do not believe plot in Matlab). Hence, the interval is complete, and from three fixed points

$$x_{*1} = -3.0644, x_{*2} = 1.8920, x_{*3} = 0.1725$$

only the third one satisfies the first Banach fixed point theorem condition. But, we are not yet sure if we can compute it iteratively as we need to fullfill two more conditions.

Exercise II

We have proven that for differentiable functions F the Lipschitz constant q in

$$\|F(\boldsymbol{x}) - F(\boldsymbol{y})\| \leq q\|\boldsymbol{x} - \boldsymbol{y}\|$$

is given as derivative of F. In our case this reads as

$$\sup \|F'(x)\| = \|\frac{3x^2 + 2x}{6}\|$$

The maximum of $F'(x)$ in K is for $x = 1$. Hence,

$$\sup_{x \in [-1,1]} |F'(x)| = \frac{5}{6} < 1$$

Hence, the function is **locally** Lipschitz continuous with the Lipschitz constant $q = \frac{5}{6}$

Exercise II

Since,

$$q = \frac{5}{6} < 1$$

one may immediately conclude that the mapping $F(x)$ is contraction on interval $K = [-1, 1]$. This further means that we can compute the root $x_{*,3}$ inside the interval $[-1, 1]$ by using the fixed point iteration and starting from any x_0 in $[-1, 1]$.

Proof of fixed point theorem

Prove

- there exists a fixed point $\mathbf{x}_*$
- $\mathbf{x}_{n+1} = F(\mathbf{x}_n)$ converges to $\mathbf{x}_*$ for any $\mathbf{x}_0 \in K$

Idea: Show that $\mathbf{x}_k$ is a Cauchy sequence for any $\mathbf{x}_0 \in K$ and then use the fact that K as a closed subset of a complete space is itself a complete.

Proof of fixed point theorem I part

Show: that for all $\epsilon > 0$ there exists an $M \in \mathbb{N}$ such that

$$\|\mathbf{x}_{k+m} - \mathbf{x}_k\| < \epsilon \qquad \text{for all } k > M \text{ and all } m > 0.$$

To show this, choose some k and $m \in \mathbb{N}$ and observe that

$$\begin{aligned}\|\mathbf{x}_{k+m} - \mathbf{x}_k\| &= \left\|\mathbf{x}_{k+m} + \sum_{j=1}^{m-1}(\mathbf{x}_{k+j} - \mathbf{x}_{k+j}) - \mathbf{x}_k\right\| \\ &= \left\|\sum_{j=1}^{m}(\mathbf{x}_{k+j} - \mathbf{x}_{k+j-1})\right\| \leq \sum_{j=1}^{m}\|\mathbf{x}_{k+j} - \mathbf{x}_{k+j-1}\|\end{aligned}$$

Here is used the triangle inequality $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$,

Proof of fixed point theorem I part

Further from $\mathbf{x}_{k+j} = F(\mathbf{x}_{k+j-1})$ one has

$$\|\mathbf{x}_{k+j} - \mathbf{x}_{k+j-1}\| = \|F(\mathbf{x}_{k+j-1}) - F(\mathbf{x}_{k+j-2})\| \leq q\|\mathbf{x}_{k+j-1} - \mathbf{x}_{k+j-2}\|$$

Also,

$$\|\mathbf{x}_{k+j-1} - \mathbf{x}_{k+j-2}\| = \|F(\mathbf{x}_{k+j-2}) - F(\mathbf{x}_{k+j-3})\| \leq q\|\mathbf{x}_{k+j-3} - \mathbf{x}_{k+j-4}\|$$

such that

$$\|\mathbf{x}_{k+j} - \mathbf{x}_{k+j-1}\| = \|F(\mathbf{x}_{k+j-1}) - F(\mathbf{x}_{k+j-2})\| \leq q^2\|\mathbf{x}_{k+j-3} - \mathbf{x}_{k+j-4}\|$$

holds.

Proof of fixed point theorem I part

Hence, by induction one may conclude that

$$\|\mathbf{x}_{k+j} - \mathbf{x}_{k+j-1}\| = \|F(\mathbf{x}_{k+j-1}) - F(\mathbf{x}_{k+j-2})\| \leq \cdots \leq q^{j-1}\|\mathbf{x}_{k+1} - \mathbf{x}_k\|$$

holds. Substituting back to the inequality

$$\|\mathbf{x}_{k+m} - \mathbf{x}_k\| \leq \sum_{j=1}^{m} \|\mathbf{x}_{k+j} - \mathbf{x}_{k+j-1}\|$$

we get

$$\|\mathbf{x}_{k+m} - \mathbf{x}_k\| \leq \sum_{j=1}^{m} \|\mathbf{x}_{k+j} - \mathbf{x}_{k+j-1}\| \leq \sum_{j=1}^{m} q^{j-1}\|\mathbf{x}_{k+1} - \mathbf{x}_k\|$$

Proof of fixed point theorem I part

This further leads to

$$\begin{aligned}\|\mathbf{x}_{k+m} - \mathbf{x}_k\| &\leq \sum_{j=1}^{m} q^{j-1} \|\mathbf{x}_{k+1} - \mathbf{x}_k\| \\ &\leq \sum_{j=1}^{\infty} q^{j-1} \|\mathbf{x}_{k+1} - \mathbf{x}_k\| = \frac{1}{1-q} \|\mathbf{x}_{k+1} - \mathbf{x}_k\|.\end{aligned}$$

Here is used the sum of geometric series:

$$\sum_{j=1}^{\infty} q^{j-1} = \frac{1}{1-q}$$

Proof of fixed point theorem I part

Also, we know that

$$\|\mathbf{x}_{k+1} - \mathbf{x}_k\| = \|F(\mathbf{x}_{k+1}) - F(\mathbf{x}_k)\| \leq q\|\mathbf{x}_k - \mathbf{x}_{k-1}\| \leq q^2\|\mathbf{x}_{k-1} - \mathbf{x}_{k-2}\|$$

which leads us to

$$\|\mathbf{x}_{k+1} - \mathbf{x}_k\| \leq q^k\|\mathbf{x}_1 - \mathbf{x}_0\|$$

With the previous slide this gives:

$$\|\mathbf{x}_{k+m} - \mathbf{x}_k\| \leq \frac{q^k}{1-q}\|\mathbf{x}_1 - \mathbf{x}_0\|$$

Because $q < 1$, q^k will decrease with increasing k, and hence one may conclude that $(\mathbf{x}_k)$ is a Cauchy sequence. Since K is complete there exists an $\mathbf{x}_* \in K$ such that $\mathbf{x}_k \longrightarrow \mathbf{x}_*$

Proof of fixed point theorem II part

Show that the fixed point is unique.

Assume that two fixed points $\mathbf{x}_* \in K$ and $\mathbf{y}_* \in K$ exist

$$\mathbf{x}_* = F(\mathbf{x}_*) \quad \text{and} \quad \mathbf{y}_* = F(\mathbf{y}_*)$$

Then

$$\|\mathbf{x}_* - \mathbf{y}_*\| = \|F(\mathbf{x}_*) - F(\mathbf{y}_*)\| \leq q\|\mathbf{x}_* - \mathbf{y}_*\|$$

Since $0 \leq q < 1$, this requires $\|\mathbf{x}_* - \mathbf{y}_*\| = 0$. Therefore, $\mathbf{x}_* = \mathbf{y}_*$ and thus the fixed point is unique.

A posteriori estimate proof

Previously we have shown that

$$\|\mathbf{x}_{k+m} - \mathbf{x}_k\| \leq \frac{1}{1-q}\|\mathbf{x}_{k+1} - \mathbf{x}_k\|$$

but also according to Banach fixed point theorem

$$\|\mathbf{x}_{k+1} - \mathbf{x}_k\| \leq q\|\mathbf{x}_k - \mathbf{x}_{k-1}\|$$

Thus,

$$\|\mathbf{x}_{k+m} - \mathbf{x}_k\| \leq \frac{q}{1-q}\|\mathbf{x}_k - \mathbf{x}_{k-1}\|$$

and as $\mathbf{x}_{k+m} \longrightarrow \mathbf{x}_*$ for $m \longrightarrow \infty$ we obtain

$$\|\mathbf{x}_* - \mathbf{x}_k\| \leq \frac{q}{1-q}\|\mathbf{x}_k - \mathbf{x}_{k-1}\|.$$

A priori error estimate proof

Having

$$\|\mathbf{x}_k - \mathbf{x}_{k-1}\| \leq q\|\mathbf{x}_{k-1} - \mathbf{x}_{k-2}\|$$

and then

$$\|\mathbf{x}_{k-1} - \mathbf{x}_{k-2}\| \leq q\|\mathbf{x}_{k-2} - \mathbf{x}_{k-3}\|$$

one may sequentially obtain

$$\|\mathbf{x}_k - \mathbf{x}_{k-1}\| \leq q\|\mathbf{x}_{k-1} - \mathbf{x}_{k-2}\| \leq \cdots \leq q^{k-1}\|\mathbf{x}_1 - \mathbf{x}_0\|.$$

From this it follows

$$\|\mathbf{x}_* - \mathbf{x}_k\| \leq \frac{q^k}{1-q}\|\mathbf{x}_1 - \mathbf{x}_0\|.$$

Speed of convergence of fixed point iteration

Until now we have defined conditions under which the fixed point iteration

$$\mathbf{x}_{k+1} = F(\mathbf{x}_k)$$

converges to a unique solution $\mathbf{x}_*$. However, sometimes more than one scheme (think about our example) are converging to $\mathbf{x}_*$ To choose the faster one, we have to study the convergence of the scheme. To do this let us expand the fixed point mapping $F(\mathbf{x})$ into Taylor series around $\mathbf{x}_*$. Let $\mathbf{d}_k := \mathbf{x}_k - \mathbf{x}_*$ be the defect/residuum.

Speed of convergence of fixed point iteration

Then,

$$\begin{aligned}\mathbf{x}_{k+1} &= F(\mathbf{x}_k) = F(\mathbf{x}_* + \mathbf{d}_k)\\ &= F(\mathbf{x}_*) + F'(\mathbf{x}_*)\mathbf{d}_k + \frac{1}{2}F''(\mathbf{x}_*)\mathbf{d}_k^2 + \mathcal{O}(|\mathbf{d}_k|^3)\\ &= \mathbf{x}_* + F'(\mathbf{x}_*)\mathbf{d}_k + \frac{1}{2}F''(\mathbf{x}_*)\mathbf{d}_k^2 + \mathcal{O}(|\mathbf{d}_k|^3)\end{aligned}$$

and

$$\mathbf{d}_{k+1} = F'(\mathbf{x}_*)\mathbf{d}_k + \frac{1}{2}F''(\mathbf{x}_*)\mathbf{d}_k^2 + \mathcal{O}(|\mathbf{d}_k|^3).$$

Convergence speed

Thus

$$|\mathbf{d}_{k+1}| \leq |F'(\mathbf{x}_*)|\,|\mathbf{d}_k| + \frac{1}{2}|F''(\mathbf{x}_*)|\,|\mathbf{d}_k|^2 + \mathcal{O}(|\mathbf{d}_k|^3).$$

If $0 < q := |F'(\mathbf{x}_*)| < 1$, the scheme converges linearly with convergence factor q in a small region around $\mathbf{x}_*$. If $|F'(\mathbf{x}_*)| = 0$, the scheme converges at least quadratically in a region around $\mathbf{x}_*$. This analysis can be generalised for the multi-dimensional case.

Convergence

Definition

Let $\mathbf{x}_*, \mathbf{x}_1, \mathbf{x}_2, \ldots \in \mathbb{R}^d$ *with* $\mathbf{x}_n \longrightarrow \mathbf{x}_*$ *for* $n \longrightarrow \infty$.

1. $\{\mathbf{x}_n\}$ *converges* *linearly with convergence factor* $q \in (0,1)$ *if*

$$\|\mathbf{x}_{n+1} - \mathbf{x}_*\| \leq q\|\mathbf{x}_n - \mathbf{x}_*\|.$$

2. $\{\mathbf{x}_n\}$ *converges* *super-linearly with order* p *if there exists a* $p > 1$ *with*

$$\|\mathbf{x}_{n+1} - \mathbf{x}_*\| \leq q\|\mathbf{x}_n - \mathbf{x}_*\|^p.$$

3. *If* $q = 0$ *and* $p = 1$, *then* $(\mathbf{x}_n)$ *converges* *super-linearly.*
4. *When* $(\mathbf{x}_n)$ *converges with order* $p = 2$, *then* $(\mathbf{x}_n)$ *is said to converge* *quadratically.*

Fixed point iteration

Problems:

- The fixed point iteration is good when the function is Lipschitz continuous and contractive on the interval containing the equilibrium point (root). Otherwise, the method diverges.
- in case of functions with more than one root, the method can miss some of them
- requires complicated algebraic transformations to obtain $F(x)$ in

$$x = F(x)$$

- etc.

Other methods?

Is there **ANY OTHER** way to solve the system

$$\mathbf{f}(\mathbf{x}_*) = \mathbf{0}$$

Answer

There are other ways, but they strongly depend if f is

- linear
- or nonlinear.