Introduction to Scientific Computing

(Lecture 3: Vector spaces and difference equations)

Bojana Rosić

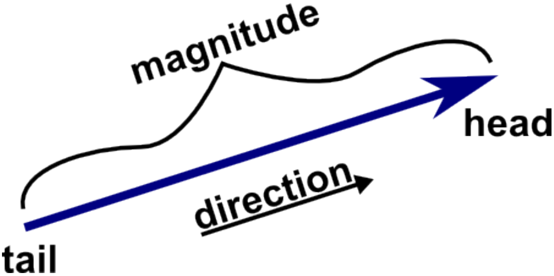
Institute of Scientific Computing

November 1, 2017

PART I: LINEAR ALGEBRA

Vector space

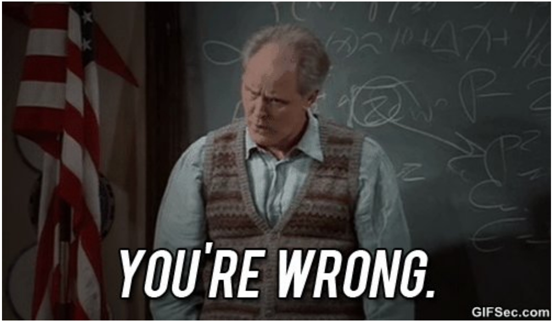
What is a vector?



http://mathinsight.org/media/image/image/vector.png

Vector space

Hmm, if that was your answer then...



Vector space

Vector space

Vector has much more general defintion:

A vector is an element of a vector space.

which somehow leads us to

Vector space

In common language the vector space V can be defined as a set of things for which the operation of addition and multiplication by scalar can be defined.

Remember that this does not include multiplication of two vectors, definition of their length etc. Example: let us have two quadratic polynomials

$$P_1 = x^2 - 1, \quad P_2 = 2x^2 + x$$

If we add them

$$P_3 = P_1 + P_2 = 3x^2 + x - 1$$

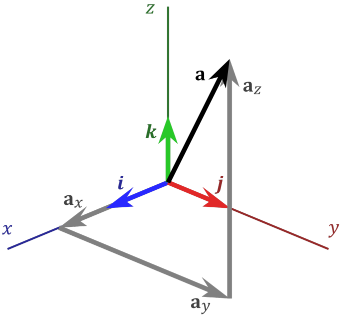
we get quadratic polynomial, too. Similarly, if we multiply them by a scalar we get again quadratic polynomial.

Vector space

For V the following axioms have to be fullfilled:

1. There is a function, addition, denoted $+$, so that $v_1 + v_2$ is another vector.
2. There is a function, multiplication by scalars, so that αv is a vector.
3. The associative law: $(v_1 + v_2) + v_3 = v_1 + (v_2 + v_3)$
4. There is a zero vector, so that for each $v, v + 0 = v$.
5. There is an additive inverse for each vector, so that for each v, there is another vector v' so that $v + v' = 0$.
6. The commutative law of addition holds: $v_1 + v_2 = v_2 + v_1$.
7. $(\alpha + \beta)v = \alpha v + \beta v$
8. $(\alpha\beta)v = \alpha(\beta v)$, 9. $\alpha(v_1 + v_2) = \alpha v_1 + \alpha v_2$, 10. $1v = v$

Typical example is Euclidian space



wiki

Examples: Real functions

Another example is the set of real-valued functions on a real variable:

$$f_1(x) + f_2(x) = f_3(x)$$

$$(\alpha f)(x) = \alpha f(x)$$

Then, the set of n tuples $(x_1, x_2, ...x_n)$ with addition and multiplication defined by

$$(x_1, x_2, ...x_n) + (y_1, y_2, ...y_n) = (x_1 + y_1, x_2 + y_2, ...x_n + y_n)$$

$$\alpha(x_1, x_2, ...x_n) = (\alpha x_1, \alpha x_2, ..., \alpha x_n)$$

makes a vector space.

Examples: Linear homogeneous equations

The system

$$\begin{aligned} x + 3y + z &= 0 \\ 4x + 2y + 2z &= 0 \end{aligned}$$

has for solution arbitrary x,and hence $y = x/2, z = -5x/2$.

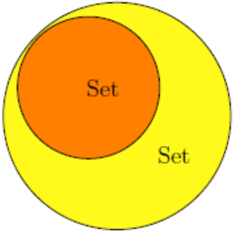
Hence, the tuple $(x, x/2, -5x/2)$ forms a vector space, as a tuple obtained by sums of two tuples $(x, x/2, -5x/2)$ is also solution of the previous system. Similarly, the tuple $(x, x/2, -5x/2)$ multipled by any scalar α would also represent the solution of the previous equation.

Subspace

Theorem

If a subset of a vector space is closed under addition and multiplication by scalars, then it is itself a vector space. This means that if you add two elements of this subset to each other they remain in the subset and multiplying any element of the subset by a scalar leaves it in the subset. It is a subspace.

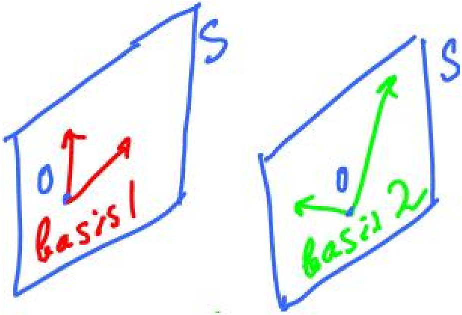
Example: a set of real valued functions on a real variable has for a subset the set of bounded functions. Also, one more example is the set of continouous functions.



mathscoop.com

Basis and dimension

The basis of a vector space is a set of linearly independent vectors (*vectors that cannot be expressed as linear combination of other vectors*) with help of which all other vectors can be expressed. The number of elements in basis defines the dimension of a vector space.



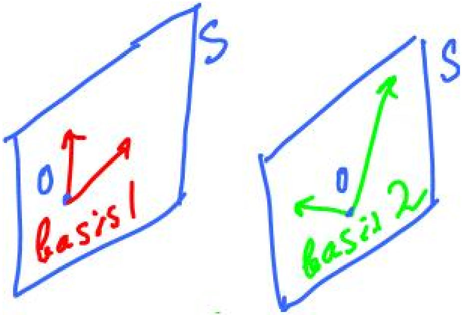
wiki

Basis and dimension

Linear independence can be tested easily by coloumnwise forming a matrix from all vectors suspected that form a basis

$$A := [\boldsymbol{v}_i]_{i=1,..,N}$$

If the determinant of the previous matrix is non-zero, then they are linearly independent.



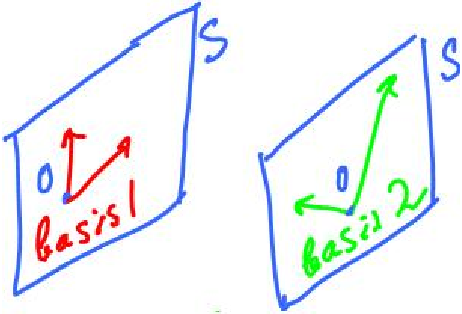
wiki

Basis and dimension

A basis is a set of vectors $\boldsymbol{v}_i$ (finite or infinite) and spans the whole space. This means that any vector from the space can be expressed as linear combination of these vectors, i.e.

$$\boldsymbol{b} = a_1 \boldsymbol{v}_1 + a_2 \boldsymbol{v}_2 + ... + a_n \boldsymbol{v}_n$$

In such as case the coefficients a_i are called the coordinates.



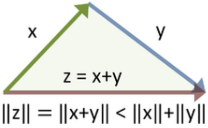
wiki

Norm

To measure the "length" of a vector v in some vector space V one uses the definition of norm.

The norm $\|\cdot\|$ is a function that can be defined on a vector space such that holds

1. $\|v\| \geq 0; \quad \|v\| = 0 \quad iff \quad v = 0$
2. $\|\alpha v\| = |\alpha| v$
3. triangle inequality
   $\|v_1 + v_2\| \leq \|v_1\| + \|v_2\|$



wiki

The norm represents "the length" of vector and can be used to compute "the distance" between v_1 and v_2:

$$\|v_1 - v_2\|$$

Norm: Euclidian space

Note that the norm is not neccessarily uniquely defined. The most known norm is Euclidean norm in the space R^n defined as

$$\|\boldsymbol{x}\| := \sqrt{x_1^2 + \cdots + x_n^2}.$$

In a special case when R^1 the norm is the same as the absolute value

$$\|\boldsymbol{x}\| := |x_1|.$$

The Euclidean norm is also called the Euclidean length, L_2 distance (or norm), ℓ_2 distance (or norm).

Norm: p-norm

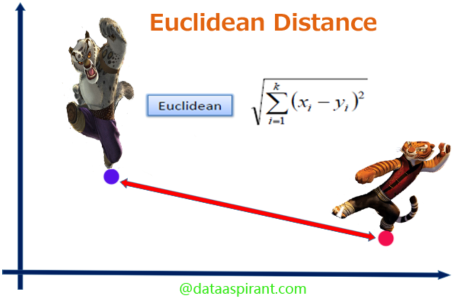
In this case one may define the generalised norm

$$\|\mathbf{x}\|_p := \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p},$$

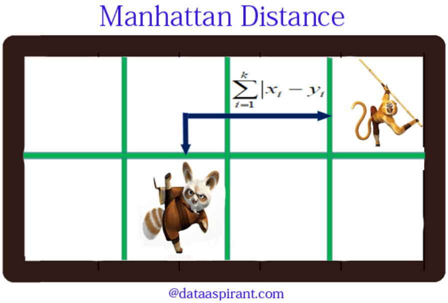
and hence distinguish different norms

- ℓ_1 (or Manhattan, taxi-) norm: $\|\boldsymbol{x}\|_1 = \sum_{i=1}^{n} |x_i|$
- ℓ_2 (or Eucledian) norm: $\|\boldsymbol{x}\|_2 = \sqrt{x_1^2 + \cdots + x_n^2}$
- ℓ_∞ norm: $\|\boldsymbol{x}\|_\infty = \max |x_i|$

Norm: *p*-norm

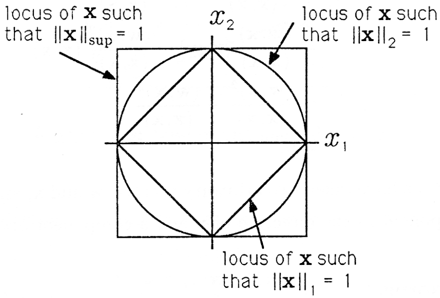


Euclidean Distance
Euclidean
$\sqrt{\sum_{i=1}^{k}(x_i - y_i)^2}$
@dataaspirant.com



Manhattan Distance
$\sum_{i=1}^{k}|x_i - y_i|$
@dataaspirant.com

Norm: p-norm



locus of x such
that ||x||sup = 1
x2
locus of x such
that ||x||2 = 1
x1
locus of x such
that ||x||1 = 1

Matrix norm

The definition of matrix norm is not so straightforward. The reason is that the matrix A can be observed at least in two different ways:

- as a vector of all its entries- elementwise norm

$$\|A\|_p = \|\mathrm{vec}(A)\|_p = \left(\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^p\right)^{1/p}$$

- as an operator acting on another vector $\boldsymbol{x}$- induced norm

$$\|A\|_p = \sup_{\boldsymbol{x}\neq o}\frac{\|A\boldsymbol{x}\|_p}{\|\boldsymbol{x}\|_p}$$

$$\|A\|_1 = \max_{1\leq j\leq n}\sum_{i=1}^{m}|a_{ij}|, \|A\|_\infty = \max_{1\leq i\leq m}\sum_{j=1}^{n}|a_{ij}|, \|A\|_2 \leq \left(\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2\right)^{1/2} = \|A\|_F\,,$$

Inner (scalar in Eucledean space) product

is a scalar valued function $\langle v_1, v_2 \rangle$ on two elements v_1 and v_2 of a vector space. This function returns scalar and satisfies the following rules

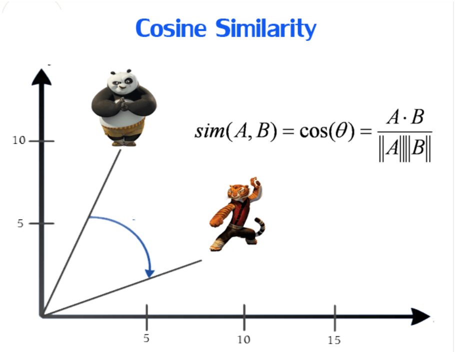
1. $\langle v_1, (v_2 + v_3) \rangle = \langle v_1, v_2 \rangle + \langle v_1, v_3 \rangle$
2. $\langle v_1, \alpha v_2 \rangle = \alpha \langle v_1 . v_2 \rangle$
3. $\langle v_1, v_2 \rangle^* = \langle v_2, v_1 \rangle$
4. $\langle v, v \rangle \geq 0; and \langle v, v \rangle = 0 \quad iff \quad v = 0$

Using scalar product (if exists) one may define $\|v\| = \sqrt{\langle v, v \rangle}$. This norm follows the Causchy-Schwartz inequality

$$|\langle v_1, v_2 \rangle| \leq \|v_1\| \cdot \|v_2\|.$$

http://www.physics.miami.edu/ nearing/mathmethods/

Angle between vectors



Complete normed vector space

A normed vector space V is called complete if every sequence $\{x_n\}_{n=1}^{\infty}$ in V converges to $x \in V$ such that

$$\lim_{n \to \infty} \|x - x_n\| = 0$$

holds.

This actually means that for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\|x - x_n\| \leq \varepsilon \quad \forall n \geq N$$

Note that this defines **convergence**. However, the defintion is not so much usable as we do not know x, and we cannot guess it.

Complete normed vector space

To be able to guess x we need to introuce the so-called Causchy sequences:

Definition

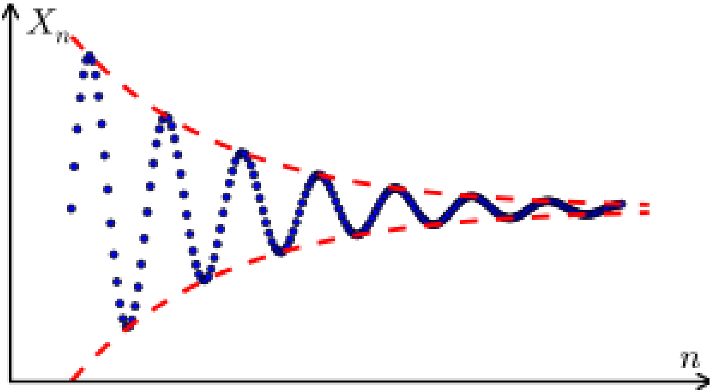
A sequence $\{x_n\}_{n=1}^{\infty}$ of real numbers is called a Cauchy sequence, if for every positive real number ϵ, there is a positive integer N such that for all natural numbers $m, n > N$ holds

$$\|x_m - x_n\| < \varepsilon.$$

Definition

A vector space V in which every Cauchy sequence converges to an element of V is called complete.

Cauchy sequence



copyright@wiki

Cauchy sequence

Let su consider the sequence

$$x_n = \frac{1}{n}$$

Let $\epsilon > 0$ be given. Then let us show that there exists an integer $N > 0$ such that $m, n > N \Rightarrow |x_m - x_n| < \epsilon$. This would be equivalent to

$$|x_m - x_n| < \varepsilon \Rightarrow \left|\frac{1}{m} - \frac{1}{n}\right| < \epsilon$$

Further more, one has

$$\left|\frac{1}{m} - \frac{1}{n}\right| < \frac{1}{m} + \frac{1}{n}$$

Cauchy sequence

To make

$$\frac{1}{m} + \frac{1}{n} < \epsilon$$

must hold

$$\frac{1}{m} < \frac{\epsilon}{2} \quad \text{and} \quad \frac{1}{n} < \frac{\epsilon}{2}$$

and hence

$$n > 2\frac{1}{\epsilon} \quad \text{and} \quad m > 2\frac{1}{\epsilon}$$

So, we see that if N is an integer larger than $2\frac{1}{\epsilon}$ then

$$m, n > N \Rightarrow |x_m - x_n| < \epsilon$$

.

Finally, the sequence $\frac{1}{n}$ is Cauchy.

Cauchy sequence

One may prove that

Definition

Every covergent sequence is a Causchy sequence.

However, does opposite holds? Does every Causchy sequence is convergent? In many spaces this is the case. These are the spaces we focus on.

Banach space

Definition

A Banach space V is a normed vector space in which each Causchy sequence $\{x_n\}_{n=1}^{\infty}$ converges to some x in V.

Thus, a Banach space V is a vector space over the set of real numbers $\mathbb{R}$ (could be complex, too) with a metric $\|\cdot\|$ that allows the computation of vector length and distance between vectors and is complete in the sense that a Cauchy sequence of vectors always converges to a well defined limit in the space. That is to say, for every Cauchy sequence x_n in V, there exists an element x in V such that

$$\lim_{n\to\infty} x_n = x.$$