Introduction to Scientific Computing

(Lecture 4: Difference equations)

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Motivation: bank account example

- balance of the bank–account after the n–th year: $x_n \in \mathbb{R}, (n \geq 0)$
- initial balance: $x_0 \in \mathbb{R}$
- rate of interest: p

Let the annual change be

$$\Delta x_n := x_{n+1} - x_n$$

such that the rate of interest can be expressed as

$$p = \frac{\Delta x_n}{x_n} \quad \Leftrightarrow \quad x_{n+1} - x_n = p x_n \Rightarrow x_{n+1} = (1 + p) x_n$$

Motivation: bank account example

The equation

$$x_{n+1} = (1 + p)x_n$$

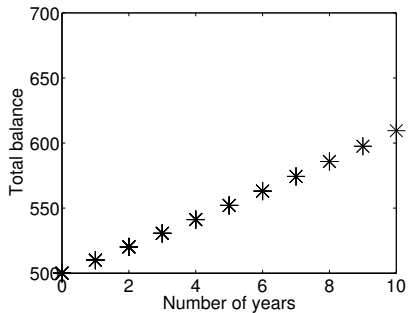is nothing else but **difference (re-curence) equation**. Namely,

$$x_1 = (1 + p)x_0$$

$$x_2 = (1 + p)x_1 = (1 + p)^2 x_0$$

etc.

Motivation: bank account example

For interest rate $p = 2\%$, initial balance $x_0 = 500$ euros and $n = 10$ years

Motivation: Fibonacci and the rabbits

Suppose a newly-born pair of rabbits, one male, one female, are put in a field. Rabbits are able to mate at the age of one month so that at the end of its second month a female can produce another pair of rabbits. Suppose that our rabbits never die and that the female always produces one new pair (one male, one female) every month from the second month on. How many pairs will there be in one year ?

Leonardo of Pisa/Fibonacci (c. 1170 – c. 1250)
copyright@wiki

Motivation: Fibonacci and the rabbits

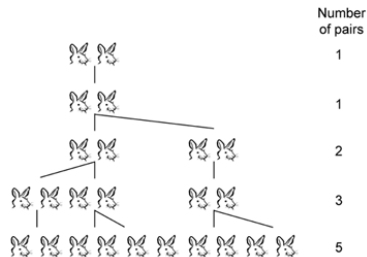This can be modelled as

$$x_n - x_{n-1} = x_{n-2}$$

or

$$x_n = x_{n-1} + x_{n-2}$$

where $x_0 = 1$ and $x_1 = 1$. The sequence by months is:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...$$

Figure 1: Fibonacci and Rabbit Breeding

http://www.landlearn.net.au/

Motivation: Fibonacci and the rabbits

For $x_0 = 1$, $x_1 = 1$ and $n = 10$ one obtains
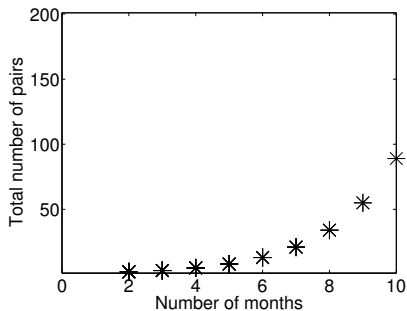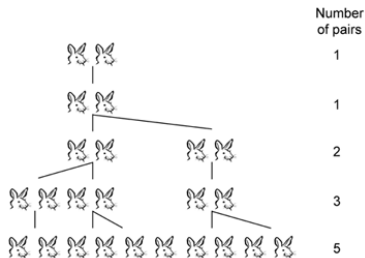

Figure 1: Fibonacci and Rabbit Breeding

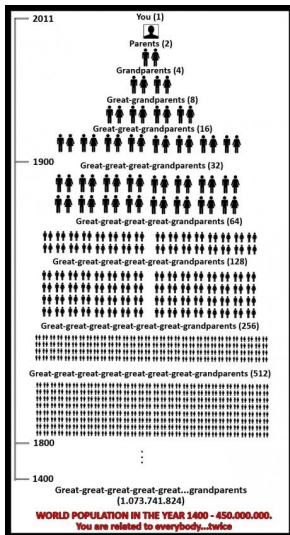http://www.landlearn.net.au/

Motivation: Logistic map

The logistic map was first proposed to describe the dynamics of insect populations. If x_n is the population of species at year n, then the simplest model of the population growth assuming that the generations are non-overlapping is given by

$$x_{n+1} - x_n = kx_n$$

in which k denotes the growth rate.

Motivation: Logistic map

However, the model could be improved by by assuming that the growth rate decreases with the population growth: saturation - less food, less space

$$k = b(c - x_n)$$

Here, c denotes the maximum capacity.

cartoonstock.com

Motivation: Logistic map
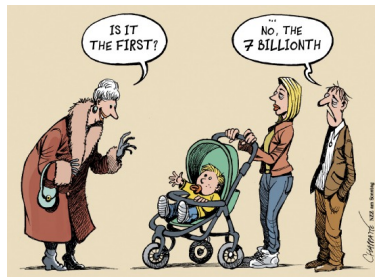
After dividing the model equation by c, one obtains

$$x_{n+1} = rx_n(1 - x_n)$$

where x_n represents the ratio of the existing populations to a possible maximum population at year n such that $0 \leq x_n \leq 1$. For example, x_0 represents the initial ratio at year 0. The parameter r is a positive number, representing the rate of reproduction and starvation.

lol-rofl.com

What are these equations?

Models

$$x_{n+1} = (1 + p)x_n$$

$$x_n = x_{n-1} + x_{n-2}$$

$$x_{n+1} = rx_n(1 - x_n)$$

are used to describe **discrete** dynamical systems, and are known as

difference equations

Difference equations

The difference equation can be written in **general (abstract)** implicit form

$$H(n, \mathbf{x}_{n+1}, \mathbf{x}_n, \ldots, \mathbf{x}_{n-k+1}) = 0,$$

in which H can be linear or nonlinear operator. Sometimes, $\mathbf{x}_{n+1}$ can be expressed in an explicit form

$$\mathbf{x}_{n+1} = F(n, \mathbf{x}_n, \mathbf{x}_{n-1}, \ldots, \mathbf{x}_{n-k+1}),$$

or

$$\Delta \mathbf{x}_n = \mathbf{x}_{n+1} - \mathbf{x}_n = G(n, \mathbf{x}_n, \mathbf{x}_{n-1}, \ldots, \mathbf{x}_{n-k+1}).$$

Note that the state $\mathbf{x}_{n+1}$ at t_{n+1} depends on the states of the k time-steps before.

Some simple examples

Examples of

$$H(n, \mathbf{x}_{n+1}, \mathbf{x}_n, \ldots, \mathbf{x}_{n-k+1}) = 0,$$

or

$$\mathbf{x}_{n+1} = F(n, \mathbf{x}_n, \mathbf{x}_{n-1}, \ldots, \mathbf{x}_{n-k+1}),$$

are

1. $x_{n+1} = nx_n \Rightarrow F = nx_n$
2. $x_{n+1} = x_n^2 \Rightarrow F = x_n^2$
3. $x_{n+1} = x_n + x_{n-2} \Rightarrow \Delta x_n = x_{n-2} \Rightarrow G = x_{n-2}$
4. $x_{n+1}^2 - x_n = n \Rightarrow H = x_{n+1}^2 - x_n - n$

What is state $\mathbf{x}_n$?

"The state is the smallest possible subset of system variables that can represent the entire state of the system at any given time"

Example:

- the water depth in the bathtub problem
- the amount of money on the bank account
- the number of pairs of rabbits, etc.

Order vs degree vs dimension

Order:

$$k = \text{highest index n} -$$
$$\text{lowest index n}$$

Degree :

$$p = \text{the highest power of}$$
$$\text{the highest indexed term}$$

Dimension :

$$d = \text{number of quantities that the state vector counts}$$

Exercise: DE of dimension 1

Order:

1. $x_{n+1} = nx_n \Rightarrow k = n + 1 - n = 1$
2. $x_{n+1} = x_n^2 \Rightarrow k = n + 1 - n = 1$
3. $x_{n+1} = x_n + x_{n-2} \Rightarrow k = n + 1 - (n - 2) = 3$
4. $x_{n+1}^2 - x_n = n \Rightarrow k = n + 1 - n = 1$

Degree:

1. $x_{n+1} = nx_n \Rightarrow p = 1$ linear
2. $x_{n+1} = x_n^2 \Rightarrow p = 1$ linear
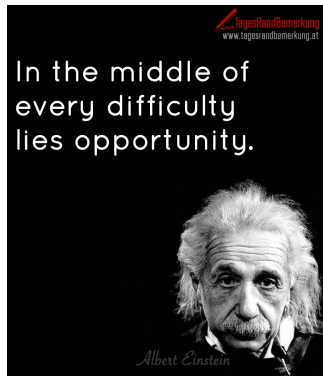3. $x_{n+1} = x_n + x_{n-2} \Rightarrow p = 1$ linear
4. $x_{n+1}^2 - x_n = n \Rightarrow p = 2$ nonlinear

Difficulty level

The higer are order and degree the more difficult is to solve!

In the middle of every difficulty lies opportunity.

Albert Einstein

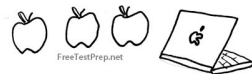# Homogeneous vs inhomogeneous

Homogeneous:

equation does not have right hand side

$$x_{n+1} - 3x_n = 0$$

Inhomogeneous :

equation has non-zero right hand side

$$x_{n+1} - 3x_n = n$$

A non-homogeneous group of apples

Autonomus vs non-autonomus

Autonomus:

equation does not depend on n

$$x_{n+1} = 3x_n$$

Non-autonomus :

equation depends on n

$$x_{n+1} = 3x_n + n$$

Exercise

Homogeneous:

1. $x_{n+1} = nx_n \Rightarrow \mathrm{yes}$
2. $x_{n+1} = x_n^2 \Rightarrow \mathrm{yes}$
3. $x_{n+1} = x_n + x_{n-2} \Rightarrow \mathrm{yes}$
4. $x_{n+1}^2 - x_n = n \Rightarrow \mathrm{no}$

Autonomus:

1. $x_{n+1} = nx_n \Rightarrow p = 1 \quad \mathrm{no}$
2. $x_{n+1} = x_n^2 \Rightarrow p = 1 \quad \mathrm{yes}$
3. $x_{n+1} = x_n + x_{n-2} \Rightarrow p = 1 \quad \mathrm{yes}$
4. $x_{n+1}^2 - x_n = n \Rightarrow p = 2 \quad \mathrm{no}$

First order difference equations (FODE)

The first order difference equation takes the form

$$x_{n+1} = F(n, x_n),$$

or

$$H(n, x_n, x_{n+1}) = 0$$

such that

$$k = n + 1 - n = 1.$$

Example: the model of the bank account

$$x_{n+1} = (1 + p)x_n$$

Solving FODE

To solve

$$x_{n+1} = F(x_n, n)$$

or

$$H(x_{n+1}, x_n, n) = 0$$

one may use

- algebraic method
- iterative (numerical) scheme

Iterative solving

To solve

$$x_{n+1} = F(x_n, n)$$

or

$$H(x_{n+1}, x_n, n) = 0$$

iteratively, one

- starts from $x_0, n = 0$
- and applies recurrence relation

$$x_{n+1} = F(x_n, n)$$

Note: take care about condition number of function F.

Iterative solving

The process of solving of

$$x_{n+1} = a x_n$$

can be done iteratively:

$$
\begin{aligned}
x_{n+1} &= a x_n \\
&= a^2 x_{n-1} \\
&= a^3 x_{n-2} \\
&= ... \\
&= a^{n+1} x_0
\end{aligned}
$$

Thus, to compute x_{n+1} one has to know **the initial condition** x_0 (the initial state).

I KNOW I CAN FIND THE ANSWER

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Algebraic solving

On the other side, one may solve

$$x_{n+1} = F(x_n, n)$$

or

$$H(x_{n+1}, x_n, n) = 0$$

directly—algebraically.

However, **note that this is not always possible!**. Thus, we will only study the process of algebraic solving linear FODE.

Linear homogeneous FODE

The difference equation

$$x_{n+1} = F(n, x_n),$$

is linear if F is linear function in x_n. In addition, it is homogeneous if

$$F(n, x_n) = a x_n$$

holds.

Example: the model of the bank account

$$x_{n+1} = (1 + p)x_n \Rightarrow a = (1 + p)$$

Algebraic solving

In case of linear homogeneous equation

$$x_{n+1} = ax_n$$

the iterative method leads us to the solution

$$x_{n+1} = a^{n+1}x_0$$

which is power function of a. This knowledge one can use to predict the solution of any linear homogeneous FODE by taking the **the ansatz**

$$x_n = c\rho^n$$

in which c denotes **the integration constant** and ρ is the **root** of the so-called characteristic equation.

Solving linear homogeneous FODE

Substituting **the ansatz**

$$x_n = c\rho^n$$

into

$$x_{n+1} = ax_n \Rightarrow x_{n+1} - ax_n = 0$$

one obtains

$$c\rho^{n+1} - ac\rho^n = 0.$$

Since $c \neq 0$ (if it is 0 then we would not have solution), one may divide the previous relation by c to obtain

$$\rho^n(\rho - a) = 0.$$

The last equation is known as **characteristic** and has the non-trivial (different than zero) solution

$$\rho = a.$$

Solving linear homogeneous FODE

Hence, the solution of

$$x_{n+1} = ax_n$$

is

$$x_n = c\rho^n = ca^n$$

In the last equation the constant c is unknown. Its estimate follows from the **initial** conditions:

$$x_0 = ca^0 \Rightarrow c = x_0$$

Finally,

$$x_n = a^n x_0$$

Examples

To compute the solution of

$$x_{n+2} - 5x_{n+1} = 0, \quad x_0 = 1$$

one may assume ansatz

$$x_n = c\rho^n.$$

Substituting the second into the first equation one obtains

$$c\rho^{n+2} - 5c\rho^{n+1} = 0$$

which leads to the **characteristic** equation (polynomial)

$$\rho - 5 = 0 \Rightarrow \rho = 5$$
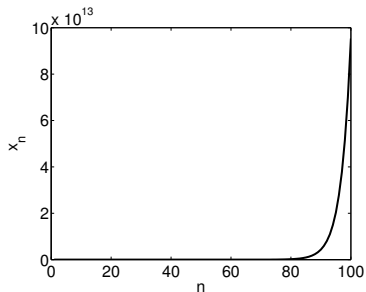
Examples

Hence,

$$x_n = c5^n$$

where c can be found from

$$x_0 = c5^0 = c \Rightarrow c = x_0 = 1$$

Finally, the solution is

$$x_n = 5^n$$

What about non-autonomous equation?

To find solution of non-autonomous equation

$$x_{n+2} - nx_{n+1} = 0, \quad x_1 = 1, n > 0$$

one may assume the same ansatz as before

$$x_n = c\rho^n$$

such that

$$\rho - n = 0$$

holds. Thus,

$$\rho = n$$

and

$$x_n = cn^n$$

What about non-autonomous equation?

From initial condition

$$x_1 = c1^1 \Rightarrow c = 1$$

such that

$$x_n = n^n$$

Check: plug-in the obtained solution $x_n = n^n$ into the original difference equation $x_{n+2} - nx_{n+1} = 0$ and check if you get identity

$$n^{n+2} - n \cdot n^{n+1} = 0 \equiv 0$$

Hence, we may conclude that $x_n = n^n$ is the solution of the difference equation $x_{n+2} - nx_{n+1} = 0$.

What about non-homogeneous equation?

To solve

$$x_{n+1} - 0.5x_n = n, \quad x_0 = 0 \tag{1}$$

one has to use a bit different approach than previously. Namely, the ansatz

$$x_n = c\rho^n$$

only satisfies the homogeneous part of the considered equation

$$x_{n+1} - 0.5x_n = 0$$

Therefore, the solution of Eq. (1) requires correction also known as the **particular** solution. Hence, the solution of Eq. (1) consists of two parts making the sum

$$x_n = x_n^{(h)} + x_n^{(p)}$$

in which $x_n^{(h)}$ denotes the homogeneous part and $x_n^{(p)}$ represents the particular part of the solution.

What about non-homogeneous equation?

The homogeneous part is the solution of

$$x_{n+1}^{(h)} - 0.5x_n^{(h)} = 0$$

and reads

$$x_n^{(h)} = c\rho^n = c0.5^n.$$

To get the general solution, we add to this one the particular solution

$$x_n^{(p)} = an + b$$

which is obtained as a **variation** of the right hand side of Eq. (1) (the method is also known as variation of constants). Finally,

$$x_n = x_n^{(h)} + x_n^{(p)} = c0.5^n + an + b$$

What about non-homogeneous equation?

Substituting

$$x_n = x_n^{(h)} + x_n^{(p)} = c0.5^n + an + b$$

into Eq. (1) one obtains

$$c0.5^{n+1} + a(n+1) + b - 0.5(c0.5^n + an + b) = n$$

Having that (because of the homogeneous solution.)

$$c0.5^{n+1} - 0.5c0.5^n \equiv 0$$

one may write

$$a(n+1) + b - 0.5an - 0.5b = n$$

What about non-homogeneous equation?

From

$$a(n+1) + b - 0.5an - 0.5b = n$$

one obtains two equations

$$0.5an = n$$

and

$$a + 0.5b = 0$$

After solving one has

$$a = 2, \quad b = -4$$

What about non-homogeneous equation?

Thus, the general solution obtains the form

$$x_n = c0.5^n + 2n - 4$$

The constant c can be obtained from the initial conditions:

$$x_0 \equiv 0 \Rightarrow x_0 = c0.5^0 + 2 \cdot 0 - 4 = c - 4 \Rightarrow c = 4$$

Finally,

$$x_n = 4 \cdot 0.5^n + 2n - 4$$

Check: plug-in the solution to original equation to obtain

$$4 \cdot 0.5^{(n+1)} + 2(n+1) - 4 - 0.5(4 \cdot 0.5^n + 2n - 4) = 2n - 2 - n + 2 = n$$

Thus, $x_n = c0.5^n + 2n - 4$ is the solution of $x_{n+1} - 0.5x_n = n$

What about non-homogeneous equation?

Let us now try to solve

$$x_{n+1} - x_n = n$$

iteratively starting from $x_0 = 0$. Thus,

$$x_1 = x_0 + 0 = 0$$

$$x_2 = x_1 + 1 = 1$$

$$x_3 = x_2 + 2 = 3$$

$$x_4 = x_3 + 3 = 6$$

$$...$$

$$x_n = \frac{1}{2}n(n-1)$$

What about non-homogeneous equation?

However, the analytical solution is not so easy to obtain. Starting from

$$x_{n+1} - x_n = n$$

we have

$$x_n = x_n^{(h)} + x_n^{(p)}$$

where $x_n^{(h)}$ is easy to obtain from the homogeneous equation

$$x_{n+1}^{(h)} - x_n^{(h)} = 0$$

This leads to

$$x_n^{(h)} = c\rho^n = c1^n = c.$$

What about non-homogeneous equation?

Similarly to before, let us assume the particular solution in a form:

$$x_n^{(p)} = an + b$$

Then substituting

$$x_n = x_n^{(h)} + x_n^{(p)} = c + an + b$$

to

$$x_{n+1} - x_n = n$$

one obtains

$$c + a(n+1) + b - c - an - b = n$$

i.e.

$$a = n$$

What about non-homogeneous equation?

Following this, our general solution would be

$$x_n = x_n^{(h)} + x_n^{(p)} = c + n^2,$$

where the integration constant c can be found from the initial condition

$$x_0 \equiv 0 \Rightarrow 0 = c + 0 \Rightarrow c = 0.$$

Hence,

$$x_n = n^2$$

But, this is not the solution we obtained by iterative algorithm!!!

What about non-homogeneous equation?

Also, if we subtitute

$$x_n = n^2$$

into

$$x_{n+1} - x_n = n$$

we obtain

$$(n+1)^2 - n^2 = n^2 + 2n + 1 - n^2 = 2n + 1 \neq n.$$

What is the problem?

Well, we have **assumed the wrong particular solution!** | *When $\rho = 1$ the particular solution* obtained by variation of constants of the right hand side

$$x_n^{(p)} = an + b$$

has to be multiplied by n, i.e.

$$x_n^{(p)} = n(an + b)$$

Re-solving

So, our new general solution is of the form:

$$x_n = x_n^{(h)} + x_n^{(p)} = c + (an + b)n$$

such that

$$x_{n+1} - x_n = n$$

turns into

$$c + (a(n + 1) + b)n - c - (an + b)n = n$$

i.e.

$$2an = n$$

and

$$a + b = 0.$$

The solution of the last two equations is

$$a = 0.5, \quad b = -0.5$$

Re-solving

Hence, the general solution is

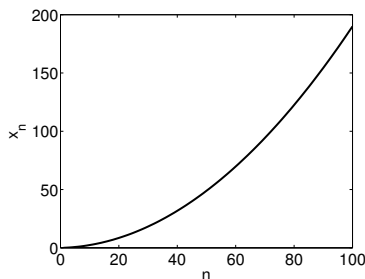$$x_n = x_n^{(h)} + x_n^{(p)} = c + \frac{1}{2}(n-1)n,$$

where the constant c can be computed from

$$x_0 = 0 \Rightarrow 0 = c + 0 \Rightarrow c = 0.$$

Finally, we obtain

$$x_n = \frac{1}{2}(n-1)n$$

which matches the iterative solution!!

Particular solution of non-homogeneous FODE

For

$$x_{n+1} = ax_n + r(n)$$

where $r(n)$ is a polynomial of order m and a is a constant, the particular solution obtains the following form

- for $a \neq 1 \Rightarrow x_n^{(p)} = b_m n^m + b_{m-1} n^{m-1} + ... + b_0$
- for $a = 1 \Rightarrow x_n^{(p)} = (b_m n^m + b_{m-1} n^{m-1} + ... + b_0)n$

For

$$x_{n+1} = ax_n + br^n$$

where a and b are constants, the particular solution is of the following form

- for $a \neq r \Rightarrow x_n^{(p)} = dr^n$
- for $a = r \Rightarrow x_n^{(p)} = dnr^n$

Linear systems of FODE (LSFODE)

The linear homogeneous system of first order difference equations is given in a form:

$$x_{n+1}^1 = a_{11}x_n^1 + a_{12}x_n^2 + ... + a_{1d}x_n^d$$

$$x_{n+1}^2 = a_{21}x_n^2 + a_{22}x_n^2 + ... + a_{2d}x_n^d$$

$$...$$

$$x_{n+1}^d = a_{d1}x_n^1 + a_{d2}x_n^2 + ... + a_{dd}x_n^d$$

which further can be written as

$$\mathbf{x}_{n+1} = A\mathbf{x}_n$$

in which $\mathbf{x}_n = [x_n^1, ..., x_n^d]^T$.

Solving

The process of solving of

$$\mathbf{x}_{n+1} = \mathbf{A}\mathbf{x}_n$$

can be done iteratively:

$$
\begin{aligned}
\mathbf{x}_{n+1} &= \mathbf{A}\mathbf{x}_n \\
&= \mathbf{A}^2\mathbf{x}_{n-1} \\
&= \mathbf{A}^3\mathbf{x}_{n-2} \\
&= ... \\
&= \mathbf{A}^{n+1}\mathbf{x}_0
\end{aligned}
$$

Thus, to compute $\mathbf{x}_{n+1}$ one has to know **the initial condition $\mathbf{x}_0$** (the initial state).

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Exercise

Small system:

1. $x_{n+1} = 0.5 x_n \Rightarrow x_n = (0.5^n) x_0$

2. for $n = 10^3$ and $x_0 = 1$

3. $x_{1001} = 9.3e - 302$

4. Elapsed time is 0.000008 seconds.

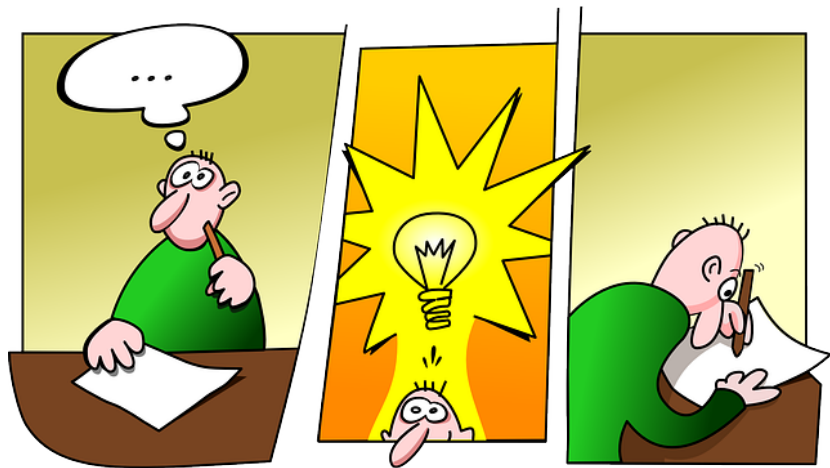Large system $\mathbf{x}_{n+1} = \mathbf{A}^{1000} \mathbf{x}_0, \quad \mathbf{x}_0 = \mathbf{1}$

1. $\mathbf{A} = \mathbb{I} \in \mathbb{R}^{2000 \times 2000}$

2. Elapsed time is 10.915375 seconds.

3. $\mathbf{A} = \mathbb{I} \in \mathbb{R}^{3000 \times 3000}$

4. Elapsed time is 34.982872 seconds.

computational time of $\mathbf{A}^{1000}$ will drastically increase with the dimension of $\mathbf{A}$!!
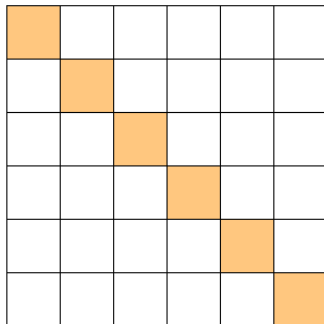
The

How to solve the problem?

Diagonalise matrix

Definition: Let $A \in \mathbb{R}^{d \times d}$ be a matrix. A is called diagonisable, if it has d linearly independent **eigenvectors** $\mathbf{v}_1, \ldots, \mathbf{v}_d \in \mathbb{C}^d$ such that

$$A\mathbf{v}_i = \lambda_i \mathbf{v}_i, \qquad i = 1, \ldots, d,$$

where $\lambda_1, \ldots, \lambda_d \in \mathbb{C}$ are the **eigenvalues** of A. Note that the eigenvalues and eigenvectors may be complex.

davidson.edu

Then it is easy...

Assumption: Let $A \in \mathbb{R}^{d \times d}$ be diagnosable. Then, its eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_d$ can be taken as a **basis** in

$$\mathbf{x}_{n+1} = A\mathbf{x}_n = A^{n+1}\mathbf{x}_0, \qquad n \in \mathbb{N},$$

such that

$$\mathbf{x}_0 = \sum_{i=1}^{d} c_i \mathbf{v}_i$$

holds. The coefficients $c_1, \ldots, c_d$ can be found by solving the linear set of equations

$$(\mathbf{v}_1 \ \mathbf{v}_2 \ \cdots \ \mathbf{v}_d) \begin{pmatrix} c_1 \\ \vdots \\ c_d \end{pmatrix} = \mathbf{x}_0.$$

Hence,..

the solution of the linear difference equation

$$\mathbf{x}_{n+1} = A\mathbf{x}_n, \qquad n \in \mathbb{N},$$

is

$$\begin{aligned}
\mathbf{x}_n = A^n \mathbf{x}_0 &= A^n \sum_{i=1}^{d} c_i \mathbf{v}_i = \sum_{i=1}^{d} c_i A^n \mathbf{v}_i = \sum_{i=1}^{d} c_i A^{n-1}[A\mathbf{v}_i] \\
&= \sum_{i=1}^{d} c_i A^{n-1} \lambda_i \mathbf{v}_i = \sum_{i=1}^{d} c_i A^{n-2} \lambda_i^2 \mathbf{v}_i = \cdots \\
&= \sum_{i=1}^{d} c_i \lambda_i^n \mathbf{v}_i.
\end{aligned}$$

How to compute eigenvalues and eigenvectors?

From

$$A\mathbf{v} = \lambda\mathbf{v}$$

one has

$$(A - \lambda I)\mathbf{v} = 0.$$

This is homogeneous system of equations which has non-trivial solution if

$$det(A - \lambda I) = 0.$$

Once λ is found, the eigenvectors can be easily computed by solving

$$(A - \lambda I)\mathbf{v} = 0.$$

Exercise

For

$$A = \begin{pmatrix} 2 & 1.5 \\ 2 & 0 \end{pmatrix}$$

one has

$$det \begin{pmatrix} 2 - \lambda & 1.5 \\ 2 & -\lambda \end{pmatrix} = 0 \Rightarrow (2 - \lambda)(-\lambda) - 3 = 0$$

Thus, the eigenvalues are

$$\lambda_1 = 3, \quad \lambda_2 = -1$$

Exercise

For corresponding eigenvector solve

$$(A - 3I)v_1 = 0, \quad (A - 3I)v_2 = 0$$

$$\begin{pmatrix} -1 & 1.5 \\ 2 & -3 \end{pmatrix} \mathbf{v}_1 = 0$$

$$-v_1^{(1)} + 1.5 v_1^{(2)} = 0$$

$$2v_1^{(1)} - 3v_1^{(2)} = 0$$

$$v_1^{(1)} = 1.5 v_2^{(1)}$$

This one has infinitely many solutions, and hence one may choose one pair $v_1^{(1)} = 3$, $v_2^{(1)} = 2$. Similarly, the second eignevector can be computed to read $(1, -2)$.
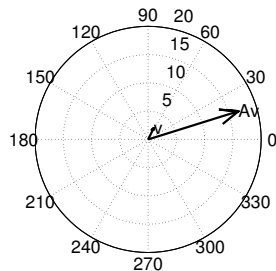
Why eigenvectors are so special?

Let us take the matrix

$$A = \begin{pmatrix} 6 & 5 \\ 1 & 2 \end{pmatrix}$$

and see what this matrix does to an arbitrary vector

$$\mathbf{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \Rightarrow A\mathbf{v} = \begin{pmatrix} 16 \\ 5 \end{pmatrix}$$

Hence, it rotates the vector as well as scales it.
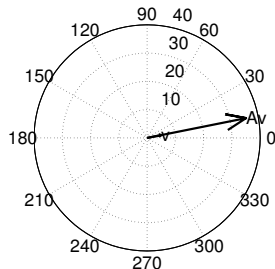

Matlab cmd: compass

Why eigenvectors are so special?

Let us take the same matrix

$$A = \begin{pmatrix} 6 & 5 \\ 1 & 2 \end{pmatrix}$$

and see what is does to its eigenvector

$$\mathbf{v} = \begin{pmatrix} 5 \\ 1 \end{pmatrix} \Rightarrow A\mathbf{v} = \begin{pmatrix} 35 \\ 7 \end{pmatrix} = 7 \begin{pmatrix} 5 \\ 1 \end{pmatrix} = 7\mathbf{v}$$


Matlab cmd: compass

Hence, it only scales the vector $\mathbf{v}$. Thus, eigenvectors cannot be **rotated** by A.
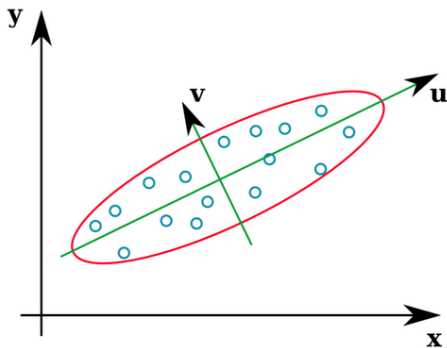
Why eigenvectors are so special?

Example: when you strech elastic band with an arrow drawn in some direction, then the arrow will preserve the direction if it represents the eigendirection (eigenvector). The eigenvalue is then the measure of its deformation.

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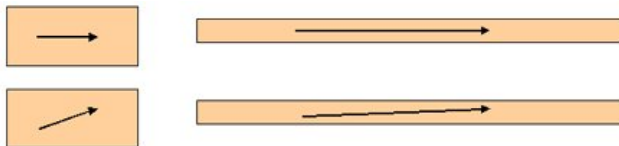
Why eigenvectors are important?

Besides making our lifes easier, eigenvectors can be used in data analysis:

In such a case one has that eigenvectors descibe directions of varying data.

Why eigenvectors are important?

In every point of the body under pressure one may define principal stress planes and principal stresses (case of no shear stress). The stress values are eigenvalues of the stress tensor, and the plane normals are the eigenvectors.
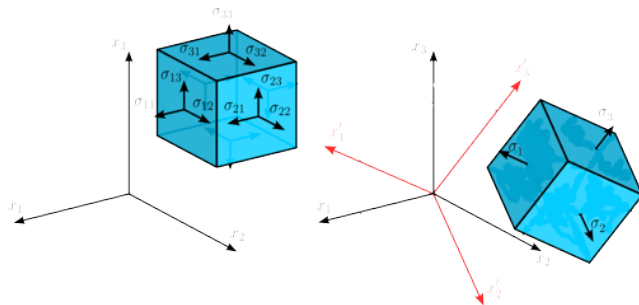
@RockMech

Back to our system

The linear difference equation

$$\mathbf{x}_{n+1} = A\mathbf{x}_n, \qquad n \in \mathbb{N},$$

has solution

$$\mathbf{x}_n = A^n \mathbf{x}_0 = A^n \sum_{i=1}^{d} c_i \mathbf{v}_i = \sum_{i=1}^{d} c_i A^n \mathbf{v}_i = \sum_{i=1}^{d} c_i A^{n-1} [A\mathbf{v}_i]$$

$$= \sum_{i=1}^{d} c_i A^{n-1} \lambda_i \mathbf{v}_i = \sum_{i=1}^{d} c_i A^{n-2} \lambda_i^2 \mathbf{v}_i = \cdots$$

$$= \sum_{i=1}^{d} c_i \lambda_i^n \mathbf{v}_i.$$

Can we always express the solution like this?

Well, the previous theory only holds when the matrix A is diagonasible which means that if

$$A \in \mathbb{R}^{d \times d}$$

then the matrix A needs to have d linearly independent vectors. Actually, this means that the matrix A has distinct eigenvalues and eigenvectors. In general case for each eigenvalue λ_i we may introduce algebraic multiplicity m_i:

$$det(A - \lambda I) = (\lambda - \lambda_1)^{m_1}(\lambda - \lambda_2)^{m_2}...(\lambda - \lambda_n)^{m_n}$$

as well as geometric multiplicity m_{gi} of λ_i as the number of corresponding eigenvectors. When the matrix is diagonasible then algebraic multiplicity of each eigenvalue and eigenvector is 1 (they do not repeat).

I case: distinct eigenvalues

Let

$$A = \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix}$$

whose eigenvalues and eigenvectors are **distinct**

$$\lambda_1 = 2.6458, \lambda_2 = -2.6458$$

$$\mathbf{v}_1 = [0.8767 \, 0.4810]^T, \mathbf{v}_2 = [-0.6354 \, 0.7722]^T.$$

Then, the solution is

$$\mathbf{x}_n = c_1 (2.6458)^n \begin{pmatrix} 0.8767 \\ 0.4810 \end{pmatrix} + c_2 (-2.6458)^n \begin{pmatrix} -0.6354 \\ 0.7722 \end{pmatrix}$$

II case: repeated eigenvalues

The matrix

$$A = \begin{pmatrix} 0 & 1 \\ -4 & 4 \end{pmatrix}$$

has **repeated** eigenvalues (algebraic multiplicity is 2)

$$\lambda := \lambda_1 = \lambda_2 = 2$$

and eigenvectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} -1 \\ -2 \end{pmatrix}.$$

Note that these two vectors are not independent because

$$\mathbf{v}_1 = -\mathbf{v}_2$$

II case: repeated eigenvalues

This further means that we have **one eigenvector and two repeated eigenvalues**. Hence, the matrix A is **deficient** and one has to search for another eigenvector in order to be able to express the solution of the previous difference equation. To do this, we will introduce the so-called Jordan's form.

Jordan form

Let us transform

$$\mathbf{x}_{n+1} = A\mathbf{x}_n$$

by

$$\mathbf{x}_n = C\mathbf{y}_n$$

to

$$\mathbf{y}_{n+1} = C^{-1}AC\mathbf{y}_n = J\mathbf{y}_n,$$

where the matrix J is of the Jordan type:

$$J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

Jordan form

Then we have that

$$CA = JC,$$

where the unknown matrix takes the form

$$C = [\mathbf{v}_1 \quad \mathbf{v}_2].$$

Its elements can be evaluated by solving the system $CA = JC$ which in component form reads

$$A\mathbf{v}_1 = \lambda\mathbf{v}_1 \Rightarrow \quad \text{the normal eigenvalue}$$

and

$$A\mathbf{v}_2 = \lambda\mathbf{v}_2 + \mathbf{v}_1 \Rightarrow \quad \text{the \textbf{generalised} eigenvalue}$$

II case: repeated eigenvalues

Once the genrealised eigenvalue $\mathbf{v}_2$ is computed, one may assume the solution in a form

$$\mathbf{x}_n = c_1 \lambda^n \mathbf{v}_1 + c_2 \lambda^n (n\mathbf{v}_1 + \mathbf{v}_2)$$

II case: repeated eigenvalues

So, in our particular example one has

$$A = \begin{pmatrix} 0 & 1 \\ -4 & 4 \end{pmatrix}$$

and

$$\lambda_1 = \lambda_2 = 2, \quad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Thus,

$$A\mathbf{v}_2 = \lambda\mathbf{v}_2 + \mathbf{v}_1$$

i.e.

$$\begin{pmatrix} 0 - 2 & 1 \\ -4 & 4 - 2 \end{pmatrix} \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

II case: repeated eigenvalues

which then produces

$$-2v_2^1 + v_2^2 = 1$$
$$-4v_2^1 + 2v_2^2 = 2$$

from which it follows one independent equation

$$-2v_2^1 + v_2^2 = 1$$

Choosing

$$v_2^1 = 0$$

one obtains

$$v_2^2 = 1$$

So, the general solution is

$$\mathbf{x}_n = c_1 2^n \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \left(n 2^n \begin{pmatrix} 1 \\ 2 \end{pmatrix} + 2^n \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$$

Higher order linear difference equations (HOLDE)

These equations have a form:

$$\sum_{i=1}^{k} a_i x_{n-i} = g_n, \quad q_i \in \mathbb{R}, \quad i = 0, ..., k,$$

in which $x_0, x_1, ... x_{k-1} \in \mathbb{R}$ are known.

Example:

$$x_{n+2} - 3x_{n+1} - 5x_{n-4} = 0$$

Solving HOLDE

Basically there are two directions you can take:

- to directly solve the problem
- to transform it to the first order system and then solve

Direct solution

Every solution $\{x_n\}$ of the inhomogeneous difference equation

$$\sum_{i=0}^{k} a_i x_{n-i} = g_n, \quad a_i \in \mathbb{R}, \ i = 0, \ldots, k$$

can be written as

$$x_n = x_n^{(h)} + x_n^{(p)}$$

where

- $x_n^{(h)}$ denotes **the homogeneous** solution
- $x_n^{(p)}$ represents **the particular** solution

Homogeneous solution

The homogeneous solution of

$$\sum_{i=0}^{k} a_i x_{n-i} = g_n, \quad a_i \in \mathbb{R}, \ i = 0, \ldots, k$$

is obtained by solving only homogeneous part
of the difference equation

$$\sum_{i=0}^{k} a_i x_{n-i}^{(h)} = 0, \quad a_i \in \mathbb{R}, \ i = 0, \ldots, k$$

Homogeneous solution ansatz

The homogeneous solution can be assumed in a form of **power** function

$$x_n^{(h)} = b\rho^n.$$

This is an educated guess (**ansatz**), which after inserting to

$$\sum_{i=0}^{k} a_i x_{n-i}^{(h)} = \sum_{i=0}^{k} a_i b \rho^{n-i} = 0$$

gives the polynomial equation

$$\chi(\rho) := \sum_{i=0}^{k} a_i \rho^{k-i} = 0$$
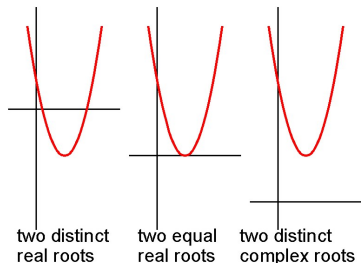
known as **characteristic** polynomial.

Homogeneous solution

The polynomial

$$\chi(\rho) := \sum_{i=0}^{k} a_i \rho^{k-i}$$

is polynomial of degree k with roots: $\rho_1, \ldots, \rho_k$.

two distinct real roots   two equal real roots   two distinct complex roots
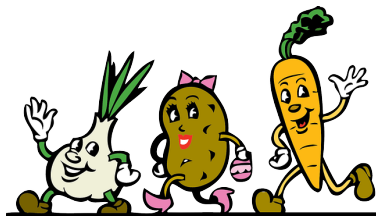
In general, the polynomial can have:

- single roots (all are different) $\rho_l \in \mathbb{R}$
- a root of multiplicity s (s times the same root) $\rho_l = \rho_{l+1} = \cdots = \rho_{l+s}$
- complex root $\rho_l \in \mathbb{C} \setminus \mathbb{R}$

Homogeneous solution for single roots

For single roots ρ_l the homogeneous solution obtains the form:

$$x_n^{(h)} = c_1 \rho_1^n + ... + c_k \rho_k^n$$

where c_l are contstants to be determined from the **initial conditions**.
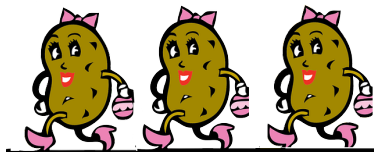
Roots with multiplicity s

If ρ_l is a root of multiplicity s, i.e.

$$\rho_l = \rho_{l+1} = \cdots = \rho_{l+s} = \rho,$$

then the corresponding homogeneous solution is

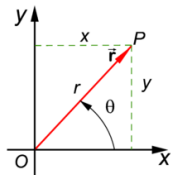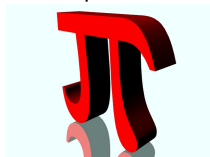$$x_n^{(h)} = c_1 \rho^n + c_2 n \rho^n ... + c_s n^{s-1} \rho^n$$

Complex roots

Let $\rho_l = \alpha + i\beta$ and $\rho_{l+1} = \alpha - i\beta$ $(\alpha, \beta \in \mathbb{R})$ be the complex root and its conjugate. To formulate the solution, let us first **transform** these roots to polar coordinates:

$$\rho_l = \mu e^{i\phi} = \mu(\cos\phi + i\sin\phi)$$
$$\rho_{l+1} = \mu e^{-i\phi} = \mu(\cos\phi - i\sin\phi),$$

where

$$\mu = |\rho_l| = |\rho_{l+1}| = \sqrt{\alpha^2 + \beta^2}, \quad \tan(\phi) = \frac{\beta}{\alpha}$$

http://en.citizendium.org/

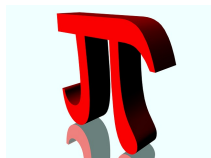are the **modulus** and the **phase**, respectively.

Complex roots

Hence, the homogeneous solution becomes:

$$x_n^{(h)} = c_1 \mu^n e^{in\phi} + c_2 \mu^n e^{-in\phi}.$$

The last relation can be rewritten in terms of trigonometrical functions:

$$x_n^{(h)} = c_1 \mu^n ((\cos n\phi + i \sin n\phi)) + c_2 \mu^n (\cos n\phi - i \sin n\phi)$$

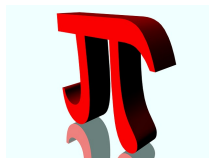$$= \mu^n (d_1 \cos n\phi + d_2 \sin n\phi)$$

Particular solution

Once the homogeneous solution $x_n^{(h)}$ of

$$\sum_{i=0}^{k} a_i x_{n-i} = 0, \quad a_i \in \mathbb{R}, \ i = 0, \ldots, k$$

is found, one may insert the general solution $x_n = x_n^{(p)} + x_n^{(h)}$ to

$$\sum_{i=0}^{k} a_i x_{n-i} = \sum_{i=0}^{k} a_i (x_{n-i}^{(h)} + x_{n-i}^{(p)}) = \sum_{i=0}^{k} a_i (x_{n-i}^{(p)}) = g_n,$$

and obtain the particular solution by solving the previous system.

Particular solution: variation of constants

Construction of $x_n^{(p)}$ depends on g_n, i. e.

g_n	$x_n^{(p)}$
a^n	$d_1 a^n$
n^k	$d_0 + d_1 n + \cdots + d_k n^k$
$n^k a^n$	$(d_0 + d_1 n + \cdots + d_k n^k) a^n$

The coefficients d_i should be choosen, such that they match the difference equation.

General solution

Finally, the general solution is obtained by summing homogeneous and particular solutions:

$$x_n = x_n^{(h)} + x_n^{(p)}$$

in which the constants c_i are found from k initial conditions (known time steps)

$$x_0 = x_0^{(h)} + x_0^{(p)}$$

$$x_1 = x_1^{(h)} + x_1^{(p)}$$

$$...$$

$$x_k = x_k^{(h)} + x_k^{(p)}$$

Exercise

To solve the difference equation

$$x_{n+2} + 2x_{n+1} + x_n = 2n, \quad x_0 = 1, x_1 = 2$$

let us first find the homogeneous solution by inserting the ansatz $x_n = b\rho^n$ to

$$x_{n+2} + 2x_{n+1} + x_n = 0.$$

In this manner we obtain the characteristic equation

$$\rho^{n+2} + 2\rho^{n+1} + \rho^n = 0$$

Exercise

whose non-trivial solutions are obtained from

$$\rho^2 + 2\rho + 1 = (\rho - (-1))^2 = 0$$

Hence, the roots are

$$\rho_{1,2} = -1$$

Thus, we have multiple root and homogeneous solution obtains the form

$$x_n^{(h)} = c_1(-1)^n + c_2 n(-1)^n$$

Exercise

Further more, the particular solution can be obtained from

$$x_{n+2}^{(p)} + 2x_{n+1}^{(p)} + x_n^{(p)} = 2n$$

by variation of constants of the right hand side:

$$x_n^{(p)} = d_0 + d_1 n$$

$$(d_0 + d_1(n+2)) + 2(d_0 + d_1(n+1)) + (d_0 + d_1 n) = 2n$$

$$4d_0 + 4d_1 = 0, \quad 4d_1 n = 2n \Rightarrow d_1 = 0.5, \quad d_0 = -0.5$$

$$x_n^{(p)} = -0.5 + 0.5n$$

Exercise

Hence, the general solution is given as:

$$x_n = x_n^{(h)} + x_n^{(p)}$$

$$x_n = c_1(-1)^n + c_2 n(-1)^n - 0.5 + 0.5n$$

The unknown constants can be found from initial conditions

$$x_0 = c_1 - 0.5 = 1 \Rightarrow c_1 = 1.5$$

and

$$x_1 = -c_1 - c_2 - 0.5 - 0.5 = 2 \Rightarrow c_2 = -1.5$$

Thus,

$$x_n = 1.5(-1)^n - 1.5n(-1)^n - 0.5 - 0.5n$$

Transformation to the FODE

The difference equation

$$\mathbf{x}_{n+1} = \mathbf{F}(n, \mathbf{x}_n, \mathbf{x}_{n-1}, \ldots, \mathbf{x}_{n-k+1}), \quad n \geq k - 1$$

where $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_{k-1} \in \mathbb{R}^d$ are given, can be **rewritten** as a first order system of dimension $d \cdot k$

$$\mathbf{y}_{m+1} = \tilde{\mathbf{F}}(m, \mathbf{y}_m), \quad m \geq 1,$$

where $\mathbf{y}_0 \in \mathbb{R}^{d \cdot k}$ is given.

Transformation to the FODE

New state vector:

$$\mathbf{y}_n = \begin{pmatrix} \mathbf{y}_n^{(1)} \\ \mathbf{y}_n^{(2)} \\ \vdots \\ \mathbf{y}_n^{(k)} \end{pmatrix} := \begin{pmatrix} \mathbf{x}_{n-k+1} \\ \mathbf{x}_{n-k+2} \\ \vdots \\ \mathbf{x}_{n-1} \end{pmatrix} \in \mathbb{R}^{d \cdot k}, \quad n \geq k-1$$

leads to

$$\mathbf{y}_{n+1} = \begin{pmatrix} \mathbf{y}_{n+1}^{(1)} \\ \mathbf{y}_{n+1}^{(2)} \\ \vdots \\ \mathbf{y}_{n+1}^{(k)} \end{pmatrix} = \begin{pmatrix} \mathbf{y}_n^{(2)} \\ \mathbf{y}_n^{(3)} \\ \vdots \\ F(n, \mathbf{y}_n^{(1)}, \mathbf{y}_n^{(2)}, \ldots, \mathbf{y}_n^{(k)}) \end{pmatrix}, \quad n \geq k$$

Transformation to the FODE

where the initial conditions become

$$\begin{pmatrix} \mathbf{y}_0^{(1)} \\ \vdots \\ \mathbf{y}_0^{(k)} \end{pmatrix} = \begin{pmatrix} \mathbf{x}_0 \\ \vdots \\ \mathbf{x}_{k-1} \end{pmatrix}.$$

Exercise

The difference equation

$$x_{n+2} + 2x_{n+1} + x_n = 2n, \quad x_0 = 1, x_1 = 2$$

can be first rewritten as

$$x_{n+1} + 2x_n + x_{n-1} = 2(n-1), \quad x_1 = 1, x_2 = 2$$

and then transformed to

$$\mathbf{y}_n = \begin{pmatrix} y_n^{(1)} \\ y_n^{(2)} \end{pmatrix} := \begin{pmatrix} x_{n-1} \\ x_n \end{pmatrix}$$

Exercise

Hence,

$$\mathbf{y}_{n+1} = \begin{pmatrix} y_{n+1}^{(1)} \\ y_{n+1}^{(2)} \end{pmatrix} = \begin{pmatrix} x_n \\ -2x_n - x_{n-1} \end{pmatrix} + \begin{pmatrix} 0 \\ 2(n-1) \end{pmatrix}$$

and thus,

$$\mathbf{y}_{n+1} = \begin{pmatrix} y_n^{(2)} \\ -2y_n^{(2)} - y_n^{(1)} \end{pmatrix} + \begin{pmatrix} 0 \\ 2(n-1) \end{pmatrix}$$

$$\mathbf{y}_{n+1} = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} \mathbf{y}_n + \mathbf{b}$$

And now you may iterate...

Literature

- Saber Elaydi, An Introduction to Difference Equations
- Arne Jensen, Lecture notes on difference equations