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Introduction to Scientific Computing: TEST EXAMPLE

Remarks:
Training exams is not a suitable preparation for the exam. Study everything.
This is a homework sheet, so it is counted as such - your points are added to your account. To achieve that this sheet has same weight as the other ones, its points are multiplied with $36/90 = 6/15 = 2/5$.

Exercise 1: *Linear ODEs* **(17 points)**

Consider the ODE

$$\ddot{x} = 2x - \dot{x}, \quad x(0) = 2, \dot{x}(0) = -1.$$

(a) Write down the analytical solution of the ODE. (12 points)

(b) How do you check stability for ordinary differential equations? (2 points)

(c) Check whether the equilibrium points are stable or not. (3 points)

Exercise 2: *Difference equations* **(28 points)**

(a) Solve the linear difference

$$x_n - 2x_{n-1} + 2x_{n-2} = 0, \qquad x_0 = 1, x_1 = 1$$

and compute x_2.

Hint: The solution of the difference equation is of the form

$$x_n = (c_1)^n \cos(n\varphi) + (c_2)^n \sin(n\varphi).$$

(16 points)

(b) How do you check stability for difference equations? (2 points)

(c) Consider the system of nonlinear difference equations given by

$$\begin{aligned} x_{n+1} &= 2 - x_n^2 + y_n, \quad x_0 = 1, \\ y_{n+1} &= 2x_n, \quad y_0 = 1. \end{aligned}$$

Compute the equilibrium points and show that the equilibrium points are unstable. (10 points)

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Exercise 3: *fixedpoint iteration* **(20 points)**

Consider the nonlinear equation

$$0.1y^3 + y = 2$$

and determine the solution of this equation by the help of the fixedpoint iteration.

(a) The solution of the problem is in the interval $[1.5, 1.7]$. Check, if the conditions of Banach's fixedpoint theorem are satisfied. (8 points)

(b) Let $y_0 = 1.55$ and compute y_1 and y_2 (4 points)

(c) Let y^* be the solution of the problem. Write down an error estimate for $|y^* - y_{20}|$ by the help of y_0 and y_1. (4 points)

(d) How can this estimate be improved if you can use y_2, too? (4 points)

Exercise 4: *Nonlinear equations* **(11 points)**

Consider the nonlinear equations

$$\begin{aligned} x_1 + x_2 &= \exp(x_1^2) \\ x_2^2 &= \exp(x_2) - 2. \end{aligned}$$

(a) Write down Newton's method for this system of equations (You need not to invert the matrix analytically). (4 points)

(b) Calculate one step of Newton's method starting from $x_0 = (0.9, 1.3)^\top$. (7 points)

Exercise 5: *Runge-Kutta methods* **(14 points)**

Consider the ODE

$$\ddot{u} - u = t^2, \quad u(0) = 1, \dot{u}(0) = 0 \tag{1}$$

and the Runge–Kutta method

$$\begin{aligned} k_1 &= f(t_m, u_m), \\ k_2 &= f(t_m + h, u_m + hk_1), \\ u_{m+1} &= u_m + \frac{h}{2}(k_1 + \beta k_2) \end{aligned}$$

(a) Write down the corresponding Butcher array for this scheme. (2 points)

(b) Choose β in such a way that the Runge–Kutta method is consistent. (3 points)

(c) Does the scheme converges? Why? (3 points)

(d) Apply the Runge–Kutta method on the ODE (1) and compute with the stepsize $h = 0.1$ a numerical approximation for $u(0.1)$. (6 points)