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Technical University Braunschweig
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Due time and date:
13:15, 14th February 2018

Introduction to PDEs and Numerical Methods (PDEs 1): Assignment 9 (50 points)

Exercise 1: *Finite elements in 1D* **(14 points)**

Consider the Poisson equation for $u : (0,2) \to \mathbb{R}$

$$-u'' = f \quad \text{on interval } (0,2)$$

with f=1 and with boundary conditions $u(0) = 0$ and $u'(2) = 3$.

(a) Derive the weak formulation of the problem and determine the trial/test space. (6 points)

$\boxed{\textit{Solution}}$

Multiplication with v, integration over interval $I = (0,2)$, and integration by parts leads to

$$-\int_I u''v = \int_I fv \tag{1}$$

$$\int_I u'v' - u'v|_0^2 = \int_I fv \tag{2}$$

where we can incorporate Neumann boundary conditions. The week form of the equation is thus:

$$\int_0^1 u'(x)v'(x)dx = \int_0^1 fvdx + 3v(2) \qquad \forall v \in V = \{f \in H^1((0,2))|v(0) = 0\}$$

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(b) The interval $[0,2]$ shall be divided into two equally sized elements of size 1 with standard linear nodal ansatz functions. Assemble the stiffness matrix $\mathbf{K}$, calculate the right hand side $\mathbf{F}$ and give the solution at the endpoints of the elements.

(8 points)

$\boxed{\textit{Solution}}$

So the global stiffness matrix will have the form:

$$\mathbf{K} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

And the right hand side will be:

$$\mathbf{F} = \begin{bmatrix} 1/2 \\ 1 \\ 3 + 1/2 \end{bmatrix}$$

The first row and first column of stiffness matrix gets cancelled because of boundary conditions. Hence we get the solution as :

$$\mathbf{u} = \begin{bmatrix} 0 \\ 4.5 \\ 8 \end{bmatrix}$$

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Exercise 2: *Finite elements in 2D* **(24 points)**

Consider the following equation for $u : \Omega \to \mathbb{R}$ defined on square domain $\Omega = (0,1)^2 \subset \mathbb{R}^2$

$$\begin{aligned} -\Delta u + u &= f \quad \text{on } \Omega \\ \mathbf{n} \cdot \nabla u &= g \quad \text{on } \partial\Omega \end{aligned}$$

with $f(\mathbf{x}) = f(\mathbf{x}) = x^2$ for $\mathbf{x} \in \Omega$ and $g(\mathbf{x}) = xy$ for $\mathbf{x} \in \partial\Omega$.

(a) Derive the weak formulation of the problem and specify the trial/test space. (6 points)

Solution

The weak formulation is

$$\int_\Omega \nabla u(x) \cdot \nabla v(x) + u(x)v(x)dx = \int_\Omega f(x)v(x)dx + \int_{\partial\Omega} g(x)v(x)ds(x) \quad \forall v \in V \tag{3}$$

where the trial/test space is

$$V = H^1(\Omega) = \{v \in L^2(\Omega) \,|\, \nabla v \in L^2(\Omega; \mathbb{R}^2)\}. \tag{4}$$

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(b) Now let's consider one specific, general rectangular element T_k. How many basis functions are nonzero over this element when you use bilinear basis functions? What is the size of the local (element) stiffness matrix of the element T_k? (3 points)

Solution

There are 4 basis functions so the size of local stiffness matrix is 4×4.

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(c) How many rows and columns the element(local)-stiffness matrix have? Give, in a general way (without any calculation), one component of the element-stiffness matrix $K_{ij}^{(e)}(x, y)$. (6 points)

Solution

The element(local)-stiffness matrix have 4 rows and 4 columns. Let's consider now the discretisation of the weak form by FEM, with nodal ansatz and basis functions $\{\phi_i\}_{i=1}^N$. Hence one component of the element stiffness matrix $K_{ij}^{(e)}(x,y)$ can be expressed as :

$$\mathbf{K}_{ij}^{(e)}(x,y) = \int_{T_k} \nabla\phi_i(\mathbf{x}) \cdot \nabla\phi_j(\mathbf{x}) dxdy,$$

with the nabla operator

$$\nabla = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix}$$

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(d) Now let's suppose, that we define the basis functions, that are nonzero over this given rectangular element, on a reference rectangle ($\hat{T}$), so we define it with a local coordinate system $\xi = [\xi \quad \eta]^T$, instead of the global coordinate system $\mathbf{x} = [x \quad y]^T$. We can easily evaluate integrals over this reference rectangle with numerical integration. However, all the differential operators and the integrals that has to be evaluated for the components of the stiffness matrix are defined on the global coordinate system. Now, let's further suppose, that you can define a mapping from the local to the global coordinates system by the mapping $F : \hat{T} \to T$ defined as:

$$F(\xi) = \mathbf{x} = \mathbf{A}\xi + \mathbf{b} \quad \mathbf{x}, \mathbf{b} \in \Re^2, \quad \mathbf{A} \in \Re^{2\times 2}$$

Or in a more detailed form:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

How do you convert the differential operators and the integration variable in $K_{ij}^{(e)}(x,y)$, that you specified in the previous task? (Here you have to give the same expression as in the previous task, but defined with the local coordinate system ξ, $K_{ij}^{(e)}(\xi,\eta)$.). (9 points)

$\boxed{\textit{Solution}}$

In global coordinate system, the stiffness matrix can be represented as :

$$\mathbf{K}_{ij}^{(e)}(x,y) = \int_{T_k} \nabla\phi_i(\mathbf{x}) \cdot \nabla\phi_j(\mathbf{x}) dxdy,$$

Using the above mentioned local coordinate system we can write the stiffness matrix as

$$\mathbf{K}_{ij}^{(e)}(\xi,\eta) = \int_{\hat{T}} \mathbf{J}^{-\mathbf{T}} \nabla_\xi \phi_i(\xi) \cdot \mathbf{J}^{-\mathbf{T}} \nabla_\xi \phi_j(\xi) |\mathbf{J}| d\xi d\eta,$$

with the local nabla operator

$$\nabla_\xi = \begin{bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta}. \end{bmatrix}$$

The change in global cordinates can be expressed using local coordinates by

$$\nabla\phi_i(\mathbf{x}) = \mathbf{J}^{-\mathbf{T}} \nabla_\xi \phi_i(\xi),$$

and the change in area of the element from global to local cordinate is given by :

$$dxdy = |\mathbf{J}| d\xi d\eta$$

number of points, n	Points, x_i	Weights, w_i
1	0	2
2	$\pm\sqrt{\frac{1}{3}}$	1
3	0	$\frac{8}{9}$
	$\pm\sqrt{\frac{3}{5}}$	$\frac{5}{9}$
4	$\pm\sqrt{\frac{3}{7}-\frac{2}{7}\sqrt{\frac{6}{5}}}$	$\frac{18+\sqrt{30}}{36}$
	$\pm\sqrt{\frac{3}{7}+\frac{2}{7}\sqrt{\frac{6}{5}}}$	$\frac{18-\sqrt{30}}{36}$
5	0	$\frac{128}{225}$
	$\pm\frac{1}{3}\sqrt{5-2\sqrt{\frac{10}{7}}}$	$\frac{322+13\sqrt{70}}{900}$
	$\pm\frac{1}{3}\sqrt{5+2\sqrt{\frac{10}{7}}}$	$\frac{322-13\sqrt{70}}{900}$

Table 1: Points and weights of the univariate Gauss-Legendre quadrature rule

where $\mathbf{J}$ is the Jacobian and $|\mathbf{J}|$ is the determinant of the jacobian which defines the ration of the areas of the original element and the one on the local coordinate system.

$$\mathbf{J} = \begin{bmatrix} \frac{\partial}{\partial \xi}x & \frac{\partial}{\partial \eta}x \\ \frac{\partial}{\partial \xi}y & \frac{\partial}{\partial \eta}y \end{bmatrix}.$$

If the map from the local to the global coordinate is defined by

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix},$$

then the Jacobian is simply the matrix $\mathbf{A}$, that is

$$\mathbf{J} = \begin{bmatrix} \frac{\partial}{\partial \xi}x & \frac{\partial}{\partial \eta}x \\ \frac{\partial}{\partial \xi}y & \frac{\partial}{\partial \eta}y \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

So the new form of the element stiffness matrix with reference to the local coordinate system is

$$\mathbf{K}_{ij}^{(e)}(\xi, \eta) = \int_{\hat{T}} \mathbf{A}^{-\mathbf{T}} \nabla_\xi \phi_i(\xi) \cdot \mathbf{A}^{-\mathbf{T}} \nabla_\xi \phi_j(\xi) |\mathbf{A}| d\xi d\eta.$$

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Exercise 3: *Numerical integration* **(12 points)**

Caclulate the integral

$$\int_{-1}^{1} f(x)dx$$

With the integrand:

$$f(x) = x^3 - 2x^2 + 5$$

using Gauß-Legendre quadrature, such way, that the result is exact. How many points are needed to get the exact integral? Calculate the solution using the table with the points and the weights in Table 1.

$\boxed{\textit{Solution}}$

with n points polynomials of order $2n-1$ can be integrated exactly, accordingly as the integrand's order is 3:

$$2n - 1 = 3 \rightarrow n = 2$$

With the two point rule:

x_i	ω_i	$f(x_i)$	$f(x_i)\omega_i$
$\sqrt{\frac{1}{3}}$	1	$\sqrt{\frac{1}{3}}^3 - \frac{2}{3} + 5$	$\sqrt{\frac{1}{3}}^3 - \frac{2}{3} + 5$
$-\sqrt{\frac{1}{3}}$	1	$-\sqrt{\frac{1}{3}}^3 - \frac{2}{3} + 5$	$-\sqrt{\frac{1}{3}}^3 - \frac{2}{3} + 5$
			$-\frac{4}{3} + 10 = 8.6667$

Therefore :

$$\int_{-1}^{1} x^3 - 2x^2 + 5dx = \sum_{i=1}^{2} f(x_i)\omega_i = f(x_1)\omega_1 + f(x_2)\omega_2 == \left(\sqrt{\frac{1}{3}}^3 - \frac{2}{3} + 5\right)\cdot 1 + \left(-\sqrt{\frac{1}{3}}^3 - \frac{2}{3} + 5\right)\cdot 1 = 8.6667$$

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