Institute of Scientific Computing
Technical University Braunschweig
Noemi Friedman, Ph.D.

Winter Term 2017
Due time and date:
13:15, 14th February 2018

Introduction to PDEs and Numerical Methods (PDEs 1): Assignment 9 (50 points)

Exercise 1: *Finite elements in 1D* **(14 points)**

Consider the Poisson equation for $u : (0, 2) \to \mathbb{R}$

$$-u'' = f \quad \text{on interval } (0, 2)$$

with f=1 and with boundary conditions $u(0) = 0$ and $u'(2) = 3$.

(a) Derive the weak formulation of the problem and determine the trial/test space. (6 points)

(b) The interval $[0, 2]$ shall be divided into two equally sized elements of size 1 with standard linear nodal ansatz functions. Assemble the stiffness matrix $\mathbf{K}$, calculate the right hand side $\mathbf{F}$ and give the solution at the endpoints of the elements.

(8 points)

Exercise 2: *Finite elements in 2D* **(24 points)**

Consider the following equation for $u : \Omega \to \mathbb{R}$ defined on square domain $\Omega = (0, 1)^2 \subset \mathbb{R}^2$

$$-\Delta u + u = f \quad \text{on } \Omega$$
$$\mathbf{n} \cdot \nabla u = g \quad \text{on } \partial\Omega$$

with $f(\mathbf{x}) = f(\mathbf{x}) = x^2$ for $\mathbf{x} \in \Omega$ and $g(\mathbf{x}) = xy$ for $\mathbf{x} \in \partial\Omega$.

(a) Derive the weak formulation of the problem and specify the trial/test space. (6 points)

(b) Now let's consider one specific, general rectangular element T_k. How many basis functions are nonzero over this element when you use bilinear basis functions? What is the size of the local (element) stiffness matrix of the element T_k? (3 points)

(c) How many rows and columns the element(local)-stiffness matrix have? Give, in a general way (without any calculation), one component of the element-stiffness matrix $K_{ij}^{(e)}(x, y)$.

(6 points)

(d) Now let's suppose, that we define the basis functions, that are nonzero over this given rectangular element, on a reference rectangle ($\hat{T}$), so we define it with a local coordinate

number of points, n	Points, x_i	Weights, w_i
1	0	2
2	$\pm\sqrt{\frac{1}{3}}$	1
3	0	$\frac{8}{9}$
	$\pm\sqrt{\frac{3}{5}}$	$\frac{5}{9}$
4	$\pm\sqrt{\frac{3}{7}-\frac{2}{7}\sqrt{\frac{6}{5}}}$	$\frac{18+\sqrt{30}}{36}$
	$\pm\sqrt{\frac{3}{7}+\frac{2}{7}\sqrt{\frac{6}{5}}}$	$\frac{18-\sqrt{30}}{36}$
5	0	$\frac{128}{225}$
	$\pm\frac{1}{3}\sqrt{5-2\sqrt{\frac{10}{7}}}$	$\frac{322+13\sqrt{70}}{900}$
	$\pm\frac{1}{3}\sqrt{5+2\sqrt{\frac{10}{7}}}$	$\frac{322-13\sqrt{70}}{900}$

Table 1: Points and weights of the univariate Gauss-Legendre quadrature rule

system $\xi = [\xi \quad \eta]^T$, instead of the global coordinate system $\mathbf{x} = [x \quad y]^T$. We can easily evaluate integrals over this reference rectangle with numerical integration. However, all the differential operators and the integrals that has to be evaluated for the components of the stiffness matrix are defined on the global coordinate system. Now, let's further suppose, that you can define a mapping from the local to the global coordinates system by the mapping $F : \hat{T} \to T$ defined as:

$$F(\xi) = \mathbf{x} = \mathbf{A}\xi + \mathbf{b} \quad \mathbf{x}, \mathbf{b} \in \Re^2, \quad \mathbf{A} \in \Re^{2\times 2}$$

Or in a more detailed form:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

How do you convert the differential operators and the integration variable in $K_{ij}^{(e)}(x, y)$, that you specified in the previous task? (Here you have to give the same expression as in the previous task, but defined with the local coordinate system ξ, $K_{ij}^{(e)}(\xi, \eta)$.). (9 points)

Exercise 3: *Numerical integration* **(12 points)**

Caclulate the integral

$$\int_{-1}^{1} f(x)dx$$

With the integrand:

$$f(x) = x^3 - 2x^2 + 5$$

using Gauß-Legendre quadrature, such way, that the result is exact. How many points are needed to get the exact integral? Calculate the solution using the table with the points and the weights in Table 1.