Institute of Scientific Computing
Technical University Braunschweig
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Winter Term 2017
Due time and date:
Wednesday, 31st January 2018

Introduction to PDEs and Numerical Methods (PDEs 1): Assignment 8 (75 points)

Exercise 1: *Galerkin method and the Finite Element Method* **(40 points)**

Consider the Poisson equation

$$-\Delta u = f$$

with $f = 2$ on the interval $[0, 1]$ with the boundary conditions $u(0) = 0$ and $u(1) = 0$.

(a) Compute the solution analytically, by direct integration. (5 points)

Solution

$$-u_{,xx} = 2$$

,

$$-\int u_{,xx}\, dx = \int 2dx$$
$$-u_{,x} = 2x + c_1$$

$$-\int u_{,x}\, dx = 2\int xdx + \int c_1 dx$$
$$u(x) = -(x^2 + c_1 x + c_2).$$

By using the boundary conditions, we get $c_2 = 0$ and $c_1 = -1$, the solution of the BVP then reads

$$u(x) = -x^2 + x.$$

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(b) Derive the weak formulation of the BVP. Identify bilinear form, linear functional, and corresponding trial and test spaces. The bilinear form of the weak equation is bounded and V-elliptic in the H_0^1 space. Why is that important? Write down an equivalent minimization problem. Knowing that the bilinear term is V-elliptic and bounded, what is the additional property of the weak form that assures this equivalence? (5 points)

Solution

By multiplying the PDE by a trial function $v(x)$ and integrating it over the domain

$$-\int_0^1 u(x)_{,xx}\, v(x)dx = \int_0^1 f(x)v(x)dx \quad \text{for } v \in V$$
$$V := H_0^1 = \Big\{v \in H^1(0, 1) | v(1) = v(0) = 0\Big\}$$

Then, integrating by parts

$$-\int_0^1 u(x)_{,xx}\, v(x)dx = \int_0^1 u_{,x}(x)v_{,x}(x) - \underbrace{u_{,x}(x)v(x)\Big|_0^1}_{0}$$

The the weak formulation of the PDE then reads the following: find $u \in V$ such that

$$\int_0^1 u_{,x}(x)v_{,x}(x) = \int_0^1 f(x)v(x)dx \quad \forall v \in V$$

And the bilinear form is

$$a(u,v) = \int_0^1 u_{,x}(x)v_{,x}(x),$$

and the linear form is

$$l(v) = \int_0^1 f(x)v(x)dx.$$

The bilinear form must be bounded and V-elliptic so that the conditions of the Lax-Milgram lemma is fulfilled (and the linear term should be also bounded); hence, there exist a unique solution $u \in V$. Suppose the functional

$$J(u) = \frac{1}{2}a(v,v) - l(v).$$

Then the equivalent minimization problem can be represented as

$$u = \arg_{v\in V} \min J(v).$$

The additional property is that $a(u,v)$ should be symmetric and positive operator.

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(c) Let's proceed now with the Galerkin method. Use both, for the basis/shape Φ_i functions and for the trial functions φ_i:

$$\Phi_i = \varphi_i = \sin(i\pi x), \quad x = 1,2,3.$$

Compute the stiffness matrix K, the right hand side of the weak form F and solve the system of equations. Make a figure with the exact solution and the one obtained by Galerkin method. (10 points)

$\boxed{\textit{Solution}}$

The discretized form of the weak form reads

$$a(u_h, \varphi_j) = l(\varphi_j) \quad j = 1,2,3,$$

with u_h being the discretized approximation of the solution u

$$u_h(x) = \sum_{i=1}^{n} u_i \Phi_i(x).$$

Which results in the system of equations

$$\sum_{i=1}^{n} u_i \underbrace{a(\Phi_i, \varphi_j)}_{=K_{ji}} = \underbrace{l(\varphi_j)}_{=F_j} \quad j = 1,2,3.$$

The diagonal elements of the stiffness matrix are

$$K_{ij} = \int_0^1 \varphi_i' \Phi_j' dx = ij\pi^2 \int_0^1 \cos(i\pi x)\cos(j\pi x)dx.$$

$$\begin{aligned}
K_{ii} &= (i\pi)^2 \int_0^1 \cos^2(i\pi x)dx \\
&= (i\pi)^2 \left[\frac{x}{2} + \frac{\sin(2i\pi x)}{4i\pi}\right]_0^1 \\
&= (i\pi)^2 \left[\frac{1}{2} + \frac{\sin(2i\pi)}{4i\pi}\right] = \frac{(i\pi)^2}{2}.
\end{aligned}$$

And the off-diagonals

$$K_{ij} = ij\pi^2 \int_0^1 \cos(i\pi x)\cos(j\pi x)dx$$

Using the identity

$$\cos(\theta)\cos(\alpha) = \frac{\cos(\theta+\alpha)+\cos(\theta-\alpha)}{2} \text{ then}$$

$$\begin{aligned}
K_{ij} &= \frac{ij\pi^2}{2}\int_0^1 \left[\cos(i\pi x + j\pi x) + \cos(i\pi x - j\pi x)\right] \\
&= \frac{ij\pi^2}{2}\left[\frac{1}{i+j}\sin\left((i+j)\pi x\right) + \frac{1}{i-j}\sin\left((i-j)\pi x\right)\right]_0^1 \\
&= \frac{ij\pi^2}{2}\left[\frac{1}{i+j}\sin\left((i+j)\pi\right) + \frac{1}{i-j}\sin\left((i-j)\pi\right)\right] = 0
\end{aligned}$$

The stiffness matrix $\mathbf{K}$ then reads

$$\mathbf{K} = \begin{bmatrix} \frac{\pi^2}{2} & 0 & 0 \\ 0 & 2\pi^2 & 0 \\ 0 & 0 & \frac{9\pi^2}{2} \end{bmatrix}$$

And the right hand side vector $\mathbf{F}$.

$$\begin{aligned}
F_j = \int_0^1 f(x)\phi_j(x)dx &= 2\int_0^1 \sin(j\pi x)dx \\
&= -\frac{2}{j\pi}\cos(j\pi x)\Big|_0^1 \\
&= -\frac{2}{j\pi}\left[\cos(j\pi) - 1\right]
\end{aligned}$$

Hence,

$$\mathbf{F} = \begin{bmatrix} \frac{4}{\pi} \\ 0 \\ \frac{4}{3\pi} \end{bmatrix}.$$

The system of equations

$$\mathbf{Ku} = \mathbf{F}$$

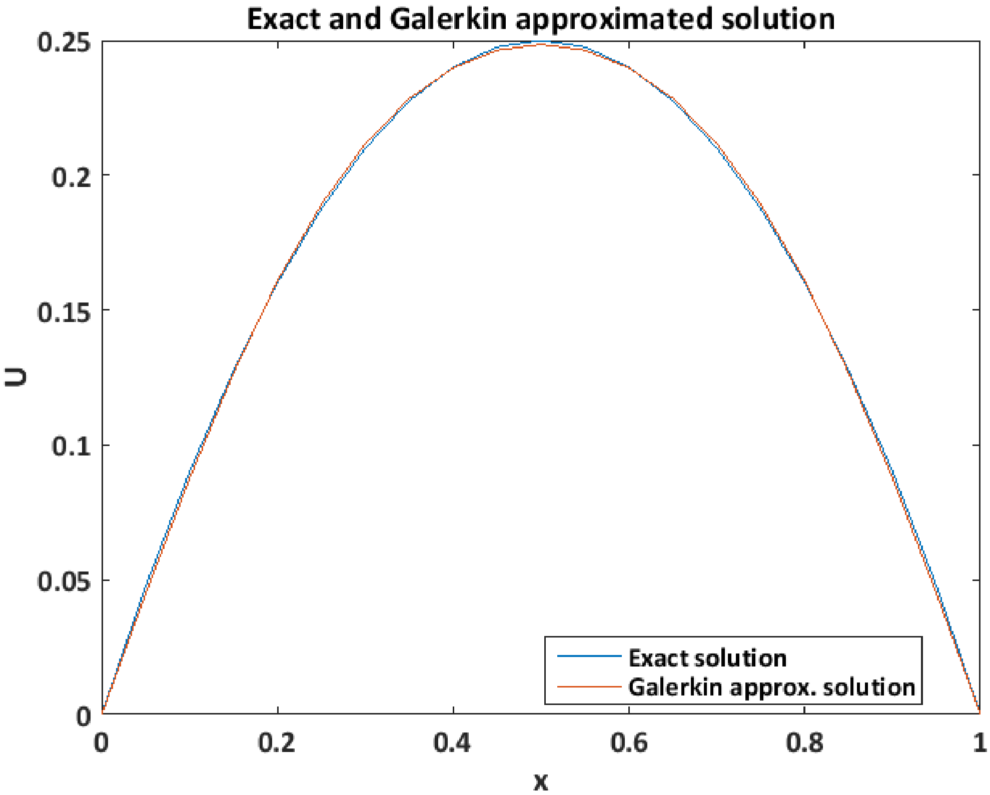
can be solved for the coefficients

$$\mathbf{u} = \begin{bmatrix} \frac{8}{\pi^3} \\ 0 \\ \frac{8}{27\pi^3} \end{bmatrix},$$

which gives the approximated solution of the PDE, which is

$$u_h(x) = \frac{8}{\pi^3}\sin(\pi x) + \frac{8}{27\pi^3}\sin(3\pi x),$$

and which is presented in the following figure.



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(d) The interval $[0, 1]$ shall be divided into three equally sized elements of size $1/3$ with standard linear nodal ansatz/shape functions. Determine and draw the shape functions. Compute the FEM solution with this mesh by hand. Assemble the stiffness matrix K, calculate the right hand side F of the weak form and solve the linear system.

(10 points)

$\boxed{\textit{Solution}}$

The shape functions are the hat functions

$$N_i = \begin{cases} \frac{x - x_{i-1}}{x_i - x_{i-1}} & x \in [x_{i-1}, x_i] \\ \frac{x_{i+1} - x}{x_{i+1} - x_i} & x \in [x_i, x_{i+1}] \\ 0 & \text{elsewhere} \end{cases}$$

The derivatives of the basis functions are

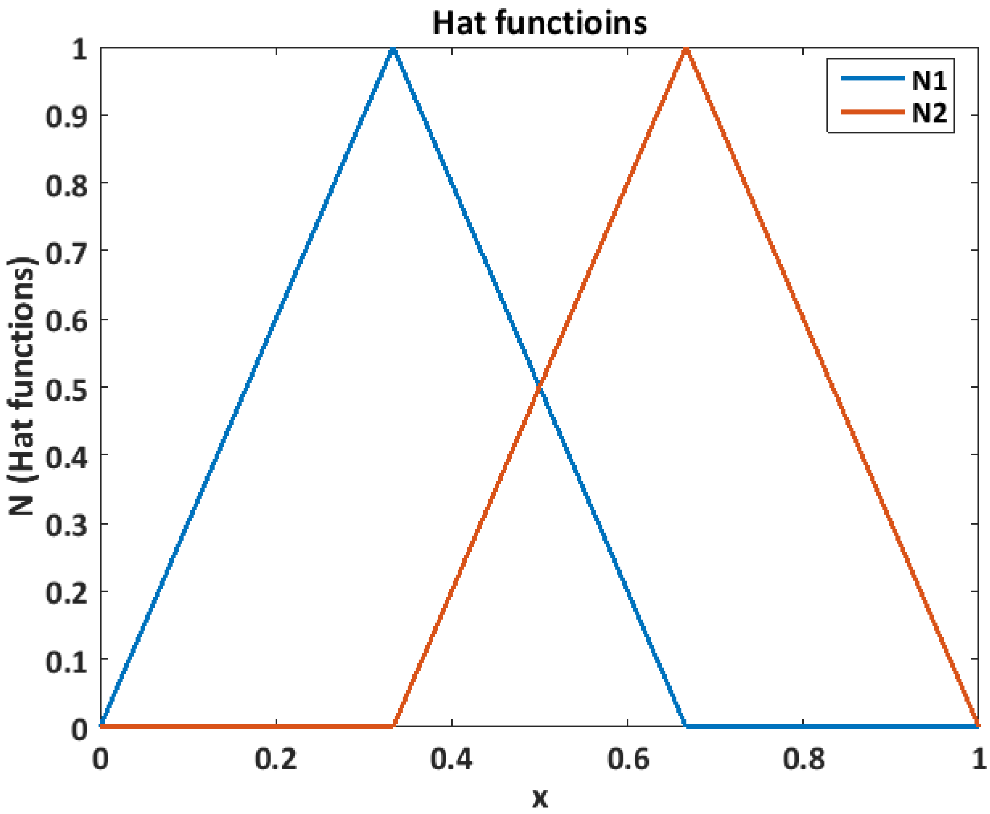
$$N_i' = \begin{cases} \frac{1}{x_i - x_{i-1}} & x \in [x_{i-1}, x_i] \\ \frac{-1}{x_{i+1} - x_i} & x \in [x_i, x_{i+1}] \\ 0 & \text{elsewhere} \end{cases}$$

For the required intervals, we got:

$$N_1 = \begin{cases} \frac{x-0}{1/3} = 3x & x \in [0, \frac{1}{3}] \\ \frac{2/3-x}{1/3} = 2-3x & x \in [\frac{1}{3}, \frac{2}{3}] \\ 0 & \text{elsewhere} \end{cases}$$

$$N_2 = \begin{cases} \frac{x-1/3}{1/3} = 3x-1 & x \in [\frac{1}{3}, \frac{2}{3}] \\ \frac{1-x}{1/3} = 3-3x & x \in [\frac{2}{3}, 1] \\ 0 & \text{elsewhere} \end{cases}$$

These hat functions are presented in the Figure below.



The new stiffness matrix reads

$$\mathbf{K} = \begin{pmatrix} \int_0^{2/3} N_1' N_1' dx & \int_{1/3}^{2/3} N_2' N_1' dx \\ \int_{1/3}^{2/3} N_2' N_1' dx & \int_{1/3}^{1} N_2' N_2' dx \end{pmatrix}.$$

where

$$\int_0^{2/3} N_1' N_1' dx = \frac{1}{l^2} x \Big|_0^{1/3} + \frac{1}{l^2} x \Big|_{1/3}^{2/3} = \frac{2}{3l^2} = 6$$

$$\int_{1/3}^{2/3} N_2' N_2' dx = \frac{1}{l^2} x \Big|_{1/3}^{2/3} + \frac{1}{l^2} x \Big|_{2/3}^{1} = \frac{2}{3l^2} = 6$$

$$\int_{1/3}^{2/3} N_2' N_1' dx = -\frac{1}{l^2} x \Big|_{1/3}^{2/3} = -\frac{1}{3l^2} = -3.$$

So the stiffness matrix is

$$\mathbf{K} = \begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix}.$$

The right hand side of the system of equations modifies to

$$\mathbf{F} = \left[\int_0^1 f(x) N_1 dx, \int_0^1 f(x) N_2 dx \right]^T$$

where

$$\int_0^1 f(x) N_1 dx = 2 \int_0^{1/3} 3x dx + 2 \int_{1/3}^{2/3} (2 - 3x) dx = \frac{2}{3}$$

$$\int_0^1 f(x) N_2 dx = 2 \int_{1/3}^{2/3} (3x - 1) dx + 2 \int_{2/3}^{1} (3 - 3x) dx = \frac{2}{3}$$

Solving the system of equations $\mathbf{Ku} = \mathbf{F}$ we get $\mathbf{u} = [2/9, 2/9]^T$, which defines the approximation of the solution

$$u_h(x) = 2/9 N_1(x) + 2/9 N_2(x),$$

which is shown in the below figure. One can observe, that due to the nodal basis, the vector $\mathbf{u}$ directly gives the nodal values of the solution.

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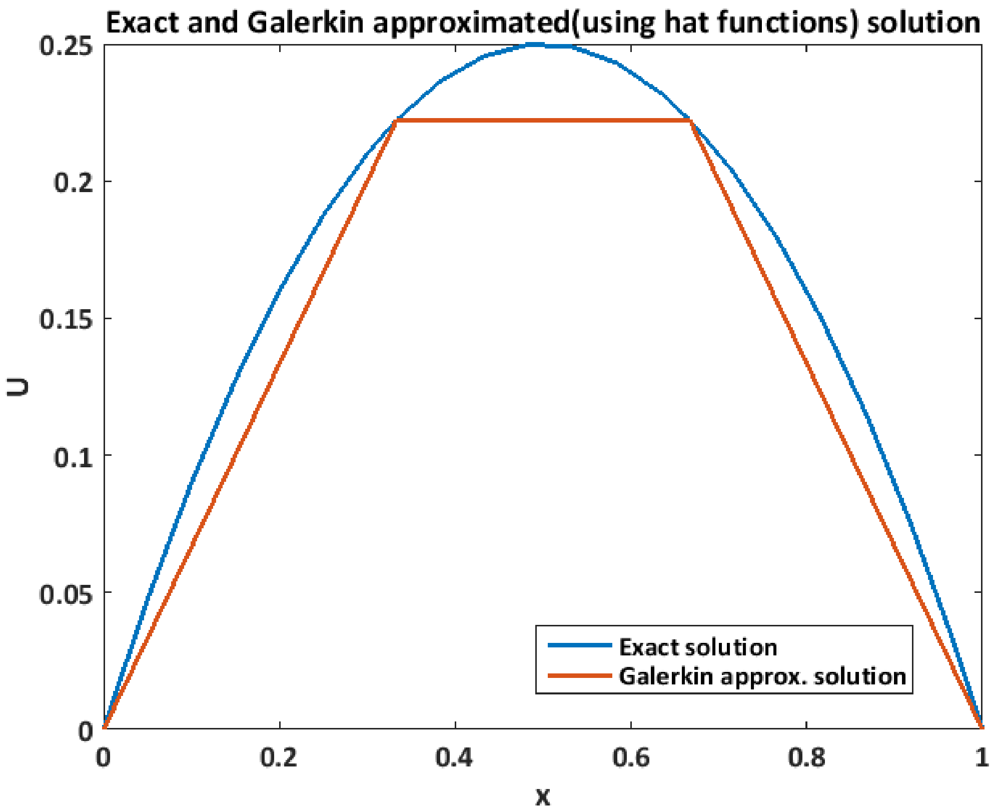
(e) Code the FEM solver of the problem in MATLAB and compare the solution with the one computed by hand. Write the code such a way, that the number of equidistant elements can be flexibly changed. Make a plot of the numerical and exact solutions. (10 points)

Solution

The solution can be observed in the following Figure



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Exercise 2: *FEM with inhomogenous Dirichlet boundary conditions* **(10 points)**

Consider again the same PDE:

$$-\Delta u = 2$$

on the interval $[0,1]$, but with inhomogenous Dirichlet Boundary Conditions $u(0) = 1$, u(1)=2.

(a) Calculate the solution analytically by direct integration. (5 points)

$\boxed{\textit{Solution}}$

$$-u_{,xx} = 2,$$

$$-\int u_{,xx}\, dx = \int 2dx$$
$$-u_{,x} = 2x + c_1$$

$$-\int u_{,x}\, dx = 2\int xdx + \int c_1 dx$$
$$u(x) = -x^2 + c_1 x + c_2$$

By using the boundary conditions, we get $c_2 = 1$ and $c_1 = 2$. The solution then reads

$$u(x) = -x^2 + 2x + 1.$$

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(b) Repeat exercise 1(d) with the given boundary conditions. (5 points)

$\boxed{\textit{Solution}}$

From the previous exercise, we know that the weak form of the equation with homogenous BC was to find $u \in V$ such that

$$a(u,v) = l(v) \quad \forall v \in V,$$

where $a(u,v) = \int_0^1 u(x)_{,x}\, v(x)$ and $l(v) = \int_0^1 f(x)v(x)dx$. The V vector space is still the same, but due to the modified inhomogeneous boundary conditions the problem changes to find $u \in V_g = \{u \in H^1(0,1), u(0) = 1, u(1) = 2\}$ such that

$$a(u,v) = l(v) \quad \forall v \in V.$$

Assuming $u(x) = w(x) + z(x)$, where $z(x)$ is any function in V_g, the weak form modifies to the following. Find $w(x) \in V$ such that

$$a(w+z,v) = l(v) \quad \forall v \in V.$$

Since $a(u,v)$ is a bilinear operator, we have: $a(w+z,v) = a(w,v) + a(z,v)$. Thus,

$$a(w,v) = l(v) - a(z,v) \quad \forall v \in V.$$

The stiffness matrix remains the same then

$$K = \begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix}.$$

For the function z we also use a discretized form, namely the ansatz using the had functions, but here we need the "half" hat functions, which are defined as:

$$N_0(x) = \begin{cases} \frac{x_1 - x}{x_1 - x_0} & 0 \leq x \leq x_1 \\ 0 & \text{elsewhere} \end{cases}$$

$$N_n(x) = \begin{cases} 0 & 0 \leq x \leq x_{n-1} \\ \frac{x - x_{n-1}}{x_n - x_{n-1}} & x_{n-1} \leq x \leq x_1 \end{cases}$$

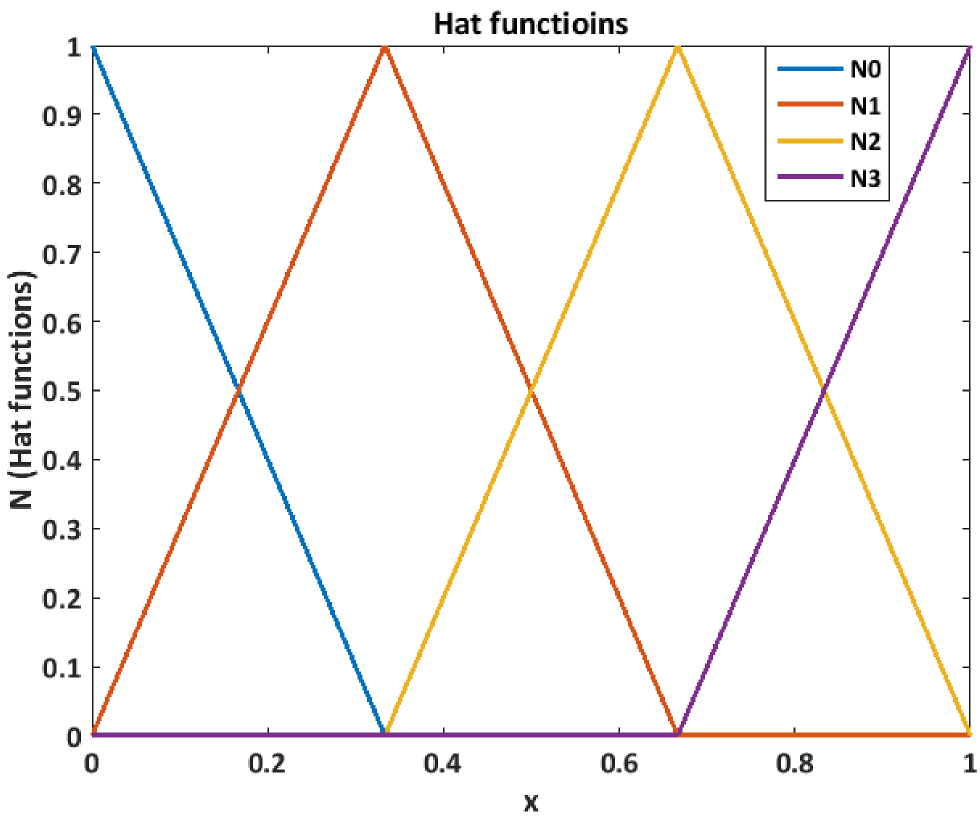
Therefore,

$$N_0'(x) = \begin{cases} \frac{-1}{x_1 - x_0} & 0 \leq x \leq x_1 \\ 0 & \text{elsewhere} \end{cases}$$

$$N_n'(x) = \begin{cases} 0 & 0 \leq x \leq x_{n-1} \\ \frac{1}{x_n - x_{n-1}} & x_{n-1} \leq x \leq x_1 \end{cases}$$

These functions together with the basis functions are visualized in the below figure. So let's take for the z function the following ansatz:

$$z(x) := \sum_{j=0}^{3} z_j N_j = u(0) \cdot N_0(x) + 0 \cdot N_1(x) + 0 \cdot N_2(x) + u(1) \cdot N_3(x) = 1 \cdot N_0(x) + 2 \cdot N_3(x).$$



The right hand side vector $\mathbf{F}$ then modifies with the term $a(z, v)$, of which the discretized form reads

$$a(z, N_i) = \sum_{j=1}^{n} z_j a(N_j, N_i) \quad i = 1, 2 \quad j = 0, 1, 2, 3$$

$$a(z,N_i)=\begin{bmatrix} a(N_0,N_1) & a(N_1,N_1) & a(N_2,N_1) & a(N_3,N_1)\\ a(N_0,N_2) & a(N_1,N_2) & a(N_2,N_2) & a(N_3,N_2)\end{bmatrix}\begin{bmatrix} z_1\\ z_2\\ z_3\\ z_4\end{bmatrix}$$

$$a(z,N_i)=\begin{bmatrix} -3 & 6 & -3 & 0\\ 0 & -3 & 6 & -3\end{bmatrix}\begin{bmatrix} u(0)\\ 0\\ 0\\ u(1)\end{bmatrix}=\begin{bmatrix} -3u(0)\\ -3u(1)\end{bmatrix}=\begin{bmatrix} -3\\ -6\end{bmatrix}$$

In the last equation we used

$$\int_0^1 N_0'N_1'dx=-\frac{1}{l^2}\int_0^{1/3}dx=-\frac{1}{3l^2}=-3$$
$$\int_0^1 N_2'N_3'dx=-\frac{1}{l^2}\int_{2/3}^{1}dx=-\frac{1}{3l^2}=-3.$$

So the new right hand side will be the right hand side vector from the previous exercise minus this modification:

$$\mathbf{F}=\begin{bmatrix}\int_0^1 f(x)N_1dx-u_0\int_0^1 N_0'N_1'dx\\ \int_0^1 f(x)N_2dx-u_n\int_0^1 N_3'N_2'dx\end{bmatrix}$$
$$\mathbf{F}=\begin{bmatrix}\frac{2}{3}\\ \frac{2}{3}\end{bmatrix}-\begin{bmatrix}-3\\ -6\end{bmatrix}=\begin{bmatrix}\frac{11}{3}\\ \frac{20}{3}\end{bmatrix}$$

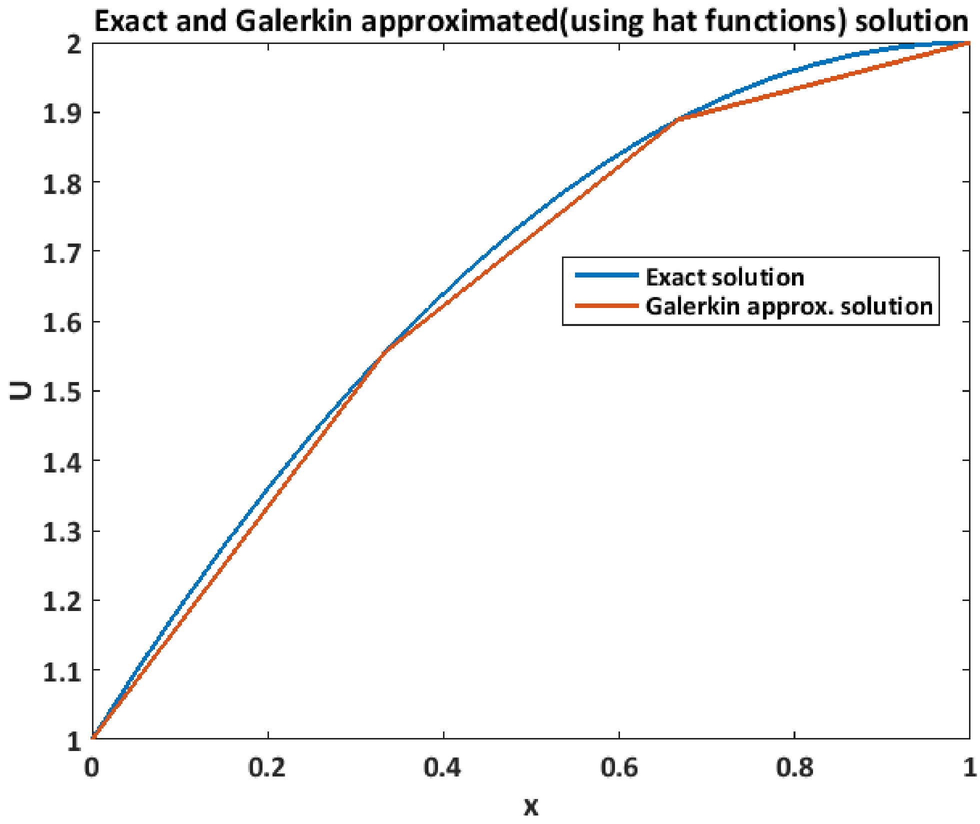
By solving the linear system of equations

$$\mathbf{Kw}=\mathbf{F} \tag{1}$$
$$\begin{bmatrix} 6 & -3\\ -3 & 6\end{bmatrix}\begin{bmatrix} w_1\\ w_2\end{bmatrix}=\begin{bmatrix}\frac{11}{3}\\ \frac{20}{3}\end{bmatrix}, \tag{2}$$

we get $\mathbf{w}=(14/9,17/9)^\top$, so the function w(x) reads $w(x)=14/9N_1(x)+17/9N_2(x)$ and the final solution reads

$$u(x)=w(x)+z(x)=14/9N_1(x)+17/9N_2(x)+1N_0(x)+2N_3(x).$$

The solution is shown in the Figure below.Because of the chosen $z(x)$ function, which in only nonzero on the inhomogenous Diriclet boundary, the values for w directly give the nodal values of the solution at the non-Dirichlet nodes.



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Exercise 3: *FEM with mixed boundary conditions* **(15 points)**

Consider again the same PDE:

$$-\Delta u = 2$$

on the interval $[0, 1]$, but with mixed boundary conditions $u(0) = 1$, $u_x(1) = 1$.

(a) Calculate the solution analytically by direct integration. (5 points)

Solution

$$-u_{,xx} = 2$$

$$-\int u_{,xx}\, dx = \int 2dx$$
$$-u_{,x} = 2x + c_1$$

where we have applied the Neumann boundary condition $c_1 = -3$. By integrating again we get

$$-\int u_{,x}\, dx = 2\int xdx - 3\int dx$$
$$u(x) = -x^2 + 3x + c_2.$$

Using the boundary condition, we get $c_2 = 1$, so the solution reads

$$u(x) = -x^2 + 3x + 1.$$

$$------$$

(b) Write down the weak formulation of the BVP. (5 points)

$\boxed{\textit{Solution}}$

By multiplying the PDE by a trial function $v(x)$ and integrating it over the domain

$$-\int_0^1 u(x)_{,xx}\, v(x)dx = \int_0^1 f(x)v(x)dx \quad \forall v \in V$$
$$V := H_0^1 = \Big\{v \in H^1(0,1) | v(0) = 0\Big\}$$

Then, integrating by parts

$$\begin{aligned} -\int_0^1 u_{,xx}(x)v(x)dx &= \int_0^1 u_{,x}(x)v_{,x}(x)dx - u(x)_{,x}\, v(x)\Big|_0^1 \\ &= \int_0^1 u_{,x}(x)v_{,x}(x)dx - u_{,x}(1)v(1) \end{aligned}$$

Therefore, the weak form of the PDE reads the following. Find $u \in V_g$ such that

$$\int_0^1 u(x)_{,x}\, v_{,x}(x)dx - v(1) = \int_0^1 f(x)v(x)dx \quad \forall v \in V$$
$$\int_0^1 u(x)_{,x}\, v_{,x}(x)dx = \int_0^1 f(x)v(x)dx + v(1) \quad \forall v \in V,$$

with $V_g = \{u \in H^1(0,1)|u(0) = 1\}$. Now we introduce again a $z(x)$ function which satisfies the inhomogeneous Dirichlet boundary condition, so a function from V_g. Such way we can write the solution function in the form $u = z + w$, so the weak form modifies to solving for $w \in V$ such that

$$\int_0^1 (w(x) + z(x))_{,x}\, v_{,x}(x)dx = \int_0^1 f(x)v(x)dx + v(1) \quad \forall v \in V$$
$$\int_0^1 w(x)_{,x}\, v_{,x}(x)dx = \int_0^1 f(x)v(x)dx + v(1) - \int_0^1 z(x)_{,x}\, v_{,x}(x)dx \quad \forall v \in V$$
$$a(w, v) = l_m(v) \quad \forall v \in V$$

with the bilinear form

$$a(w, v) = \int_0^1 w(x)_{,x}\, v_{,x}(x)dx$$

and the linear form

$$l_m(v) = \int_0^1 f(x)v(x)dx + v(1) - \int_0^1 z(x)_{,x}\, v_{,x}(x)dx.$$

To conclude, our bilinear form did not change, only the linear form on the right hand side.

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(c) Repeat exercise 1(d) with the given boundary conditions. (5 points)

$\boxed{\textit{Solution}}$

First we fix our $z(x) \in V_g$ function. Let that be

$$z(x) = u(0)N_0 = N_0.$$

The system of equation must be redefined, as we have to use the basis function N_3 too for approximating the $w(x)$ function. The stiffness matrix then modifies too a 3 by 3 matrix

$$\mathbf{K} = \begin{bmatrix} \int_0^{2/3} N_1' N_1' dx & \int_{1/3}^{2/3} N_2' N_1' dx & 0 \\ \\ \int_{1/3}^{2/3} N_2' N_1' dx & \int_{1/3}^{1} N_2' N_2' dx & \int_{2/3}^{1} N_2' N_3' dx \\ \\ 0 & \int_{2/3}^{1} N_2' N_3' dx & \int_{2/3}^{1} N_3' N_3' dx \end{bmatrix}.$$

with $\int_{2/3}^{1} N_2' N_3' dx = -3$ and $\int_{2/3}^{1} N_3' N_3' dx = 3$. Then,

$$\mathbf{K} = \begin{bmatrix} 6 & -3 & 0 \\ -3 & 6 & -3 \\ 0 & -3 & 3 \end{bmatrix}.$$

The right hand side reads

$$\mathbf{F} = \begin{bmatrix} \int_0^1 f(x) N_1 dx \\ \int_0^1 f(x) N_2 dx \\ \int_0^1 f(x) N_3 dx \end{bmatrix} + \begin{bmatrix} N_1(1) \\ N_2(1) \\ N_3(1) \end{bmatrix} - \begin{bmatrix} u(0) \int_0^1 N_0' N_1' dx \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{F} = \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} -3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{11}{3} \\ \frac{2}{3} \\ \frac{4}{3}. \end{bmatrix}$$

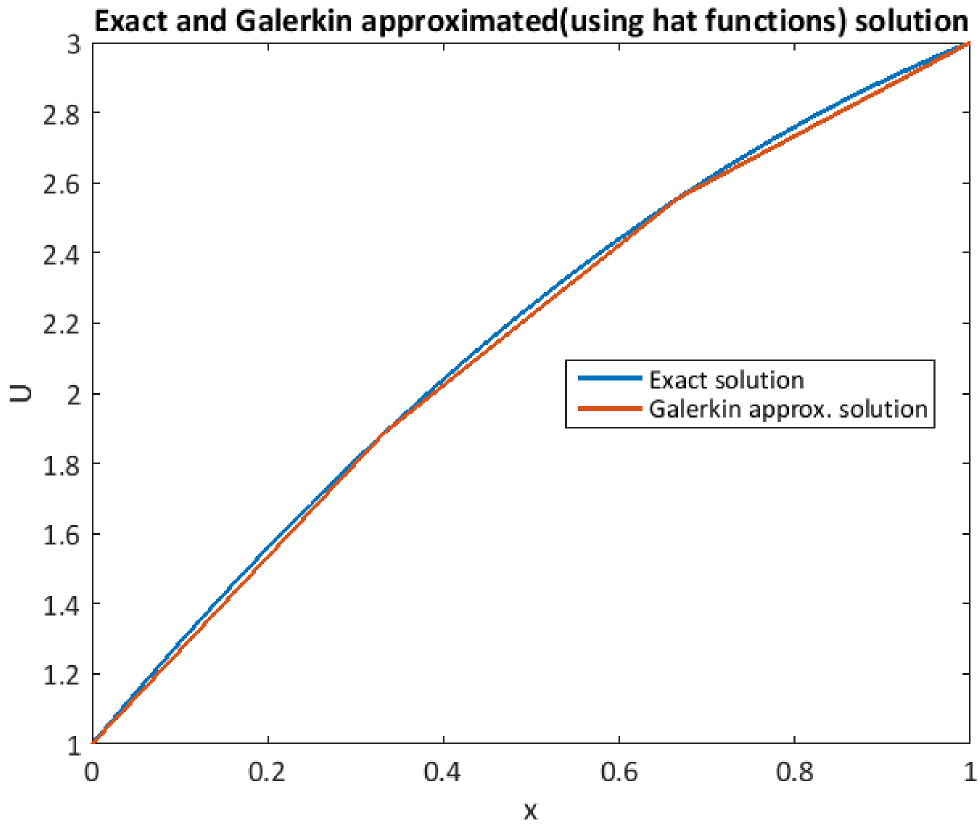
To conclude, our task is to solve the system of equations

$$\begin{bmatrix} 6 & -3 & 0 \\ -3 & 6 & -3 \\ 0 & -3 & 3 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} \frac{11}{3} \\ \frac{2}{3} \\ \frac{4}{3}. \end{bmatrix}$$

Which solution is $\mathbf{w} = [w_1, w_2, w_3]^T = [17/9, 23/9, 3]^T$. The solution function is then computed by adding $w(x)$ and $z(x)$

$$u(x) = w(x) + z(x) = 17/9 N_1(x) + 23/9 N_2(x) + 3 N_3(x) + 1 N_0(x).$$

This solution in shown in the next Figure. Because of the chosen $z(x)$ function, which in only nonzero on the inhomogenous Diriclet boundary, the values for w directly give the nodal values of the solution at the non-Dirichlet nodes.



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Exercise 4: *FEM with Neumann boundary conditions* **(10 points)**

Consider now the same PDE:

$$-\Delta u = 2$$

on the interval $(0, 1)$, but with the Neumann boundary conditions $u_x(0) = 1$, $u_x(1) = -1$.

(a) Find the analytical solution by direct integration. How many solutions do you get? (5 points)

Solution

$$-u_{,xx} = 2$$

$$-\int u_{,xx}\, dx = \int 2dx$$
$$-u_{,x} = 2x + c_1$$

Applying the Neumann boundary condition, we get $c_1 = -1$. By integrating again

$$-\int u_{,x}\, dx = 2\int x dx - \int 1dx$$
$$u(x) = -x^2 + x + c_2$$

Then, $u(x) = -x^2 + x + c_2$ for arbitrary $c \in \mathbb{R}$. Thus, there are infinitely many solutions.

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(b) What will be changed if you assume homogeneous Neumann boundary conditions $u'(0) = u'(1) = 0$. Is there a solution? (5 points)

Solution

$$-u_{,xx} = 2$$

$$-\int u_{,xx}\,dx = \int 2dx$$
$$-u_{,x} = 2x + c_1$$

Apply the Neumann boundary condition.

$$u'(0) = 0 \qquad c_1 = 0.$$
$$u'(1) = 0 \qquad c_1 = -2.$$

Thus we can conclude that there is no solution when we have homogenous Neumann boundary conditions.

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