Institute of Scientific Computing
Technical University Braunschweig
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Winter Term 2017
Due time and date:
Wednesday, 31st January
2018

Introduction to PDEs and Numerical Methods (PDEs 1): Assignment 8 (75 points)

Exercise 1: *Galerkin method and the Finite Element Method* **(40 points)**

Consider the Poisson equation

$$-\Delta u = f$$

with $f = 2$ on the interval $[0, 1]$ with the boundary conditions $u(0) = 0$ and $u(1) = 0$.

(a) Compute the solution analytically, by direct integration. (5 points)

(b) Derive the weak formulation of the BVP. Identify bilinear form, linear functional, and corresponding trial and test spaces. The bilinear form of the weak equation is bounded and V-elliptic in the H_0^1 space. Why is that important? Write down an equivalent minimization problem. Knowing that the bilinear term is V-elliptic and bounded, what is the additional property of the weak form that assures this equivalence? (5 points)

(c) Let's proceed now with the Galerkin method. Use both, for the basis/shape Φ_i functions and for the trial functions φ_i:

$$\Phi_i = \varphi_i = \sin(i\pi x), \quad x = 1, 2, 3.$$

Compute the stiffness matrix K, the right hand side of the weak form F and solve the system of equations. Make a figure with the exact solution and the one obtained by Galerkin method. (10 points)

(d) The interval $[0, 1]$ shall be divided into three equally sized elements of size 1/3 with standard linear nodal ansatz/shape functions. Determine and draw the shape functions. Compute the FEM solution with this mesh by hand. Assemble the stiffness matrix K, calculate the right hand side F of the weak form and solve the linear system.

(10 points)

(e) Code the FEM solver of the problem in MATLAB and compare the solution with the one computed by hand. Write the code such a way, that the number of equidistant elements can be flexibly changed. Make a plot of the numerical and exact solutions. (10 points)

Exercise 2: *FEM with inhomogenous Dirichlet boundary conditions* **(10 points)**

Consider again the same PDE:

$$-\Delta u = 2$$

on the interval $[0, 1]$, but with inhomogenous Dirichlet Boundary Conditions $u(0) = 1$, u(1)=2.

(a) Calculate the solution analytically by direct integration. (5 points)

(b) Repeat exercise 1(d) with the given boundary conditions. (5 points)

Exercise 3: *FEM with mixed boundary conditions* **(15 points)**

Consider again the same PDE:

$$-\Delta u = 2$$

on the interval $[0, 1]$, but with mixed boundary conditions $u(0) = 1$, $u_x(1) = 1$.

(a) Calculate the solution analytically by direct integration. (5 points)

(b) Write down the weak formulation of the BVP. (5 points)

(c) Repeat exercise 1(d) with the given boundary conditions. (5 points)

Exercise 4: *FEM with Neumann boundary conditions* **(10 points)**

Consider now the same PDE:

$$-\Delta u = 2$$

on the interval $(0, 1)$, but with the Neumann boundary conditions $u_x(0) = 1$, $u_x(1) = -1$.

(a) Find the analytical solution by direct integration. How many solutions do you get? (5 points)

(b) What will be changed if you assume homogeneous Neumann boundary conditions $u'(0) = u'(1) = 0$. Is there a solution? (5 points)