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Introduction to PDEs and Numerical Methods (PDEs 1): Assignment 1 (50 points)

Exercise 1: *Differential operators* **(15 points)**

(a) Let $f_1(x, y, z) = x^2 e^{(-3y)} \cos(2z)$. Determine $\frac{\partial f_1}{\partial x}$, $\frac{\partial f_1}{\partial y}$, $\frac{\partial f_1}{\partial z}$ and ∇f_1. (4 points)

Solution (a)

$$\frac{\partial f_1}{\partial x} = 2xe^{(-3y)} \cos(2z) \quad \frac{\partial f_1}{\partial y} = -3x^2 e^{(-3y)} \cos(2z) \quad \frac{\partial f_1}{\partial z} = -2x^2 e^{(-3y)} \sin(2z)$$

$$\nabla f_1(x, y) = \begin{bmatrix} 2xe^{(-3y)} \cos(2z) \\ -3x^2 e^{(-3y)} \cos(2z) \\ -2x^2 e^{(-3y)} \sin(2z) \end{bmatrix}$$

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(b) Let $\mathbf{f_2}(x, y, z) = (\cos(xy), xy, e^{(2z)})^T$. Determine $\nabla \cdot \mathbf{f_2}$ and $\nabla \times \mathbf{f_2}$. (4 points)

Solution

$$\nabla \cdot \mathbf{f_2} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix}^T \begin{bmatrix} \cos(xy) \\ xy \\ e^{(2z)} \end{bmatrix} = \frac{\partial \cos(xy)}{\partial x} + \frac{\partial (xy)}{\partial y} + \frac{\partial e^{(2z)}}{\partial z} = -y \sin(xy) + x + 2e^{(2z)}$$

$$\nabla \times \mathbf{f_2} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \times \begin{bmatrix} \cos(xy) \\ xy \\ e^{(2z)} \end{bmatrix} = \begin{bmatrix} \frac{\partial e}{\partial y}^{(2z)} - \frac{\partial xy}{\partial z} \\ -\frac{\partial e^{(2z)}}{\partial x} + \frac{\partial \cos(xy)}{\partial z} \\ \frac{\partial xy}{\partial x} - \frac{\partial \cos(xy)}{\partial y} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ y + x \sin(xy) \end{bmatrix}$$

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(c) Determine Δf_1 (see the function f_1 in subtask (a)). (3 points)

Solution

$\Delta f_1 = \frac{\partial^2 f_1}{\partial x^2} + \frac{\partial^2 f_1}{\partial y^2} + \frac{\partial^2 f_1}{\partial z^2}$.

$\frac{\partial^2 f_1}{\partial x^2} = 2e^{-3y}\cos(2z)$

$\frac{\partial^2 f_1}{\partial y^2} = 9x^2 e^{-3y}\cos(2z)$

$\frac{\partial^2 f_1}{\partial z^2} = -4x^2 e^{-3y}\cos(2z)$

Therefore, $\Delta f_1 = 2e^{-3y}\cos(2z) + 9x^2 e^{-3y}\cos(2z) - 4x^2 e^{-3y}\cos(2z) = e^{-3y}\left((2+5x^2)\cos(2z)\right)$.

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(d) Show that $\nabla \cdot \nabla f = \Delta f$ and $\nabla \times \nabla f = 0$ for any two-times differentiable function $f : \Omega \to \mathbb{R}^3$. (4 points)

Solution

$\nabla \cdot \nabla f = \frac{\partial}{\partial x}\frac{\partial f}{\partial x} + \frac{\partial}{\partial y}\frac{\partial f}{\partial y} + \frac{\partial}{\partial z}\frac{\partial f}{\partial z} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \Delta f$

Using Matrix-vector Notation:

$\nabla \times \nabla f = \frac{\partial}{\partial x_i}\boldsymbol{e_i} \times \frac{\partial f}{\partial x_j}\boldsymbol{e_j} = \frac{\partial^2 f}{\partial x_i x_j}\epsilon_{ijk}\boldsymbol{e_k} = (\frac{\partial^2 f}{\partial x_2 x_3} - \frac{\partial^2 f}{\partial x_3 x_2})\boldsymbol{e_1} + (\frac{\partial^2 f}{\partial x_3 x_1} - \frac{\partial^2 f}{\partial x_1 x_3})\boldsymbol{e_2} + (\frac{\partial^2 f}{\partial x_1 x_2} - \frac{\partial^2 f}{\partial x_2 x_1})\boldsymbol{e_3}$

Since $\frac{\partial^2 f}{\partial x_i x_j} = \frac{\partial^2 f}{\partial x_j x_1}$, the sum for every one of the vector components is zero.

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Exercise 2: *Heat equation* **(17 points)**

Consider the heat equation on a bar of unit length, with parameter $\beta^2 = \frac{\lambda}{\rho c}$:

$$\frac{\partial}{\partial t}\theta(x,t) - \beta^2 \frac{\partial^2}{\partial x^2}\theta(x,t) = f(x,t)$$

(a) Assume boundary conditions $\theta(0,t) = 0$, $\theta(\pi,t) = 0$ and the source term $f(x,t) = \sin(x)$. Prove that $\theta(x,t) = \sin(x)$ can be a solution of the heat equation and specify the value of β that ensures this proof.

Solution

First let's check if $\theta(x,t) = \sin(x)$ fulfills the boundary conditions:

$$\theta(0,t) = 0$$

$$\theta(\pi,t) = 0$$

Then let's check if $\theta(x,t) = \sin(x)$ is a solution to the heat equation. The left hand side (lhs) of the equation is:

$$\frac{\partial}{\partial t}\theta(x,t) - \beta^2 \frac{\partial^2}{\partial x^2}\theta(x,t) = 0 + \beta^2 \sin(x) = \beta^2 \sin(x)$$

And the right hand (rhs) side is:

$$f(x) = \sin(x)$$

As the lhs and the rhs only equal if $\beta = \pm 1$ (from which only $\beta = 1$ makes physically sense), so it is proved that $\theta(x,t) = \sin(x)$ is a solution of the given initial boundary value problem if $\beta = 1$.

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(5 points)

(b) Now assume $\beta^2 = 4$, boundary conditions $\theta(0,t) = \theta(1,t) = 0$ and a solution $\theta(x,t) = (t^2 + \frac{1}{2})\sin(\pi x)$. What must $f(x,t)$ look like if the heat equation should be satisfied.

Solution

First let's check if $\theta(x,t) = (t^2 + \frac{1}{2})\sin(\pi x)$ fulfills the boundary conditions:

$$\theta(0,t) = (t^2 + \frac{1}{2})0 = 0$$

$$\theta(1,t) = (t^2 + \frac{1}{2})0 = 0$$

$$\frac{\partial}{\partial t}\theta(x,t) - 4\frac{\partial^2}{\partial x^2}\theta(x,t) = 2t\sin(\pi x) + 4\pi^2(t^2 + \frac{1}{2})\sin(\pi x) = (4t^2\pi^2 + 2t + 2\pi^2)\sin(\pi x)$$

$$f(x,t) = (4t^2\pi^2 + 2t + 2\pi^2)\sin(\pi x)$$

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(7 points)

(c) Prove that $\theta(x,t) = t + \frac{1}{2}x^2$ is a solution of the heat equation. Write down the corresponding boundary and initial conditions.

Solution

$$\frac{\partial}{\partial t}\theta(x,t) - \beta^2\frac{\partial^2}{\partial x^2}\theta(x,t) = 1 - \beta^2 = f(x,t)$$

$\theta(x,t) = t + \frac{1}{2}x^2$ is only a solution if $f(x,t) = 1 - \beta^2$ and the boundary conditions are:

$$\theta(0,t) = t$$

$$\theta(1,t) = (t + \frac{1}{2})$$

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(5 points)

Exercise 3: *Classification of differential equations* **(8 points)**

Classify (order, linear/nonlinear, stationary/instationary, homogeneous, inhomogeneous) the following differential equations:

(a)

$$\frac{\partial^3 u}{\partial x^3} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} = 0$$

(4 points)

Solution

4th Order. Since it does not depend on time, it is stationary. Consider $u = 0$, and replace it in the Differential Equation, then:

$\frac{\partial^3 u}{\partial x^3} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} = 0$. Therefore, the equation is homogeneous.

Linearity condition: $L(\alpha u + \beta v) = \alpha L(u) + \beta L(v)$. In this case, $L(u) = \frac{\partial^3 u}{\partial x^3} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4}$.

Then, $L(\alpha u + \beta v) = \frac{\partial^3 (\alpha u + \beta v)}{\partial x^3} + 2\frac{\partial^4 (\alpha u + \beta v)}{\partial x^2 \partial y^2} + \frac{\partial^4 (\alpha u + \beta v)}{\partial y^4}$

$= \alpha\frac{\partial^3 u}{\partial x^3} + \beta\frac{\partial^3 v}{\partial x^3} + 2\alpha\frac{\partial^4 u}{\partial x^2 \partial y^2} + 2\beta\frac{\partial^4 v}{\partial x^2 \partial y^2} + \alpha\frac{\partial^4 u}{\partial y^4} + \beta\frac{\partial^4 v}{\partial y^4}$

$= \alpha(\frac{\partial^3 u}{\partial x^3} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4}) + \beta(\frac{\partial^3 v}{\partial x^3} + 2\frac{\partial^4 v}{\partial x^2 \partial y^2} + \frac{\partial^4 v}{\partial y^4}) = \alpha L(u) + \beta L(v)$.

Consequently, the PDE is linear.

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(b)

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + \sin(u) = x\sin(t)$$

(4 points)

Solution

2nd Order, instationary PDE. Since u = 0 leads to a non-zero RHS value $x\sin(t)$, it is a non-homogeneous PDE.

Linearity condition: $L(\alpha u + \beta v) = \alpha L(u) + \beta L(v)$.

In this case, $L(u) = \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + \sin(u)$

Then, $L(\alpha u + \beta v) = \frac{\partial^2 (\alpha u + \beta v)}{\partial t^2} - \frac{\partial^2 (\alpha u + \beta v)}{\partial x^2} + \sin(\alpha u + \beta v)$

$= \alpha(\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2}) + \beta(\frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 v}{\partial x^2}) + \sin(\alpha u)\cos(\beta v) + \cos(\alpha u)\sin(\beta v) \neq \alpha L(u) + \beta L(v)$.

The term leading to non-linearity is $\sin(u)$

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Exercise 4: *Classification of differential equations 2* **(10 points)**

(a) Determine and sketch the subsets of $\mathbb{R}^2$, where the following equations are elliptic/-parabolic/hyperbolic:

$$u_{xx} + 2u_x + (1 - y^2)u_{yy} + u = 0$$

Solution

$$A = 1$$
$$B = 0$$
$$C = (1 - y^2)$$
$$AC - B^2 = (1 - y^2) \to \begin{array}{ll} |y| < 0 & \text{elliptic} \\ y = 0 & \text{parabolic} \\ |y| > 0 & \text{hyperbolic} \end{array}$$

So the equation is parabolic in the point $y = 0$, elliptic in the points $|y| < 0$ and hyperbolic for $|y| > 0$.

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(5 points)

(b) Determine whether the following equations are elliptic, parabolic or hyperbolic:

$$u_{xx} - u_{xy} + 2\ u_y + u_{yy} - 3\ u_{yx} + 4u = 0$$

Solution

$$A = 1$$
$$2B = -4$$
$$C = 1$$
$$AC - B^2 = (1 - 4) = -3 < 0 \to \text{hyperbolic}$$

So the equation is hyperbolic.

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$$9\ u_{xx} + 6\ u_{xy} + u_{yy} + u_x = 0$$

Solution

$$A = 9$$
$$2B = 6$$
$$C = 1$$
$$AC - B^2 = (9 - 9) = 0 \to \text{parabolic}$$

To conclude, the equation is parabolic.

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(5 points)