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Winter Term 2016
Due time and date:
Monday, 12. December 2016

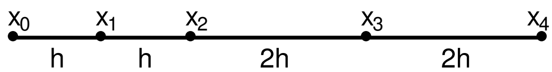
Introduction to PDEs and Numerical Methods (PDEs 1): Assignment 5 (60 points)

Exercise 1: *FDM and Irregular Meshes* **(14 points)**

After a spatial discretisation using the Finite Difference method the instationary heat equation can be expressed as

$$\dot{u}(t) = Au(t) + f$$

where matrix A encapsulates the second order differences. Until now, we have considered a regular mesh, that means, the discrete spatial points were chosen equidistantly. Now, we switch to irregular meshes, which are commonly not used when applying FDM. In this exercise you shall answer the question, why irregular meshes do not seem to be attractive here. Consider the following discrete mesh:



Dirichlet conditions are given by $u_0(t) = u_4(t) = 0$

(a) The formulas for approximating $\frac{\partial^2 u}{\partial x^2}(x_1)$ and $\frac{\partial^2 u}{\partial x^2}(x_3)$ can be taken from the script (e.g. pp 17-18). Use two Taylor expansions around x_2 and sum them up in a weighted manner to derive a formula approximating $\frac{\partial^2 u}{\partial x^2}(x_2)$.

Solution

The formulas for approximating the second derivatives in the x_1 and x_2 points:

$$\frac{\partial^2 u}{\partial x^2}(x_1) = \frac{1}{h^2}(u(x_0) - 2u(x_1) + u(x_2)) + O(h^2)$$

$$\frac{\partial^2 u}{\partial x^2}(x_3) = \frac{1}{4h^2}(u(x_2) - 2u(x_3) + u(x_4)) + O(h^2)$$

Derivation of the formula for approximating the second derivatives in the x_2 point:

$$u(x_2 + 2h) = u(x_2) + 2\frac{\partial u}{\partial x}(x_2)h + \frac{4}{2}\frac{\partial^2 u}{\partial x^2}(x_2)h^2 + \frac{8}{3!}\frac{\partial^3 u}{\partial x^3}(x_2))h^3 + O(h^4) \tag{1}$$

$$u(x_2 - h) = u(x_2) - \frac{\partial u}{\partial x}(x_2)h + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(x_2)h^2 - \frac{1}{3!}\frac{\partial^3 u}{\partial x^3}(x_2)h^3 + O(h^4) \tag{2}$$

Multiplying eq. (2) by two and adding eq. (1):

$$u(x_2 + 2h) + 2u(x_2 - h) = 3u(x_2) + 3\frac{\partial^2 u}{\partial x^2}(x_2)h^2 + \frac{6}{3!}\frac{\partial^3 u}{\partial x^3}(x_2)h^3 + O(h^4) =$$

$$= 3u(x_2) + 3\frac{\partial^2 u}{\partial x^2}(x_2)h^2 + O(h^3)$$

Rearranging the equation and dividing by h^2 gives the second derivative:

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2}(x_2) &= \frac{2u(x_2 - h) - 3u(x_2) + u(x_2 + 2h)}{3h^2} + O(h) \\ &= \frac{2u(x_1) - 3u(x_2) + u(x_3)}{3h^2} + O(h)\end{aligned}$$

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What can you say about the order of the error of your formula compared to the one of a regular mesh?

Solution

The formula is only a first order one instead of the second order approximation of the regular mesh.

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(8 points)

(b) Build up matrix A for the given irregular mesh. What can be noticed compared to the matrix resulting from a regular mesh?

Solution

$$\begin{bmatrix}\dot{u}_1(t) \\ \dot{u}_2(t) \\ \dot{u}_3(t)\end{bmatrix} = -\frac{\beta^2}{h^2}\begin{bmatrix}2 & -1 & 0 \\ -2/3 & 1 & -1/3 \\ 0 & -1/4 & 1/2\end{bmatrix}\begin{bmatrix}u_1(t) \\ u_2(t) \\ u_3(t)\end{bmatrix}$$

So the matrix is:

$$\mathbf{A} = -\frac{\beta^2}{h^2}\begin{bmatrix}2 & -1 & 0 \\ -2/3 & 1 & -1/3 \\ 0 & -1/4 & 1/2\end{bmatrix}$$

Which is a nonsymmetric matrix (while the one of the regular mesh was symmetric)

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(6 points)

Exercise 2: *Neumann stability analysis* **(31 points)**

Apply the von Neumann stability analysis for checking the stability of the following scheme for the instationary heat equation:

$$\frac{u_{j,n+1} - u_{j,n}}{\Delta t} = \frac{\beta^2}{2(\Delta x)^2}\left(u_{j-1,n+1} + u_{j-1,n} - 2u_{j,n+1} - 2u_{j,n} + u_{j+1,n+1} + u_{j+1,n}\right)$$

(a) Determine the gain factor $G(k)$ for the given scheme.

Solution

The given scheme corresponds to the Theta method with $\theta = 1/2$, and consequently the gain factor can be straightforwardly calculated from the script, by plugging in $\theta = 1/2$ to the eq. (1.137). For a better understanding we derive here the gain factor from the start.

Inserting the Ansatz:

$$u_{j,n} = G(k)^n e^{ikj\Delta x}$$

to the scheme and dividing by $G(k)^n e^{ikj\Delta x}$ gives:

$$\frac{G(k)-1}{\Delta t} = \frac{\beta^2}{2(\Delta x)^2}\left(G(k)e^{-ik\Delta x} + e^{-ik\Delta x} - 2G(k) - 2 + G(k)e^{+ik\Delta x} + e^{+ik\Delta x}\right)$$

Using the formula:

$$\cos\xi = \frac{1}{2}\left(e^{i\xi} + e^{-i\xi}\right)$$

and rearranging the equation gives:

$$2\left(G(k)-1\right) = rG(k)\left(2\cos(k\Delta x) - 2\right) + 2r\cos(k\Delta x) - 2r)$$

with

$$r = \frac{\beta^2 \Delta t}{(\Delta x)^2}$$

Solving for $G(k)$, we get the following expression for the gain factor:

$$G(k) = \frac{1 - r(1-\cos(k\Delta x))}{1 + r(1-\cos(k\Delta x))}$$

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(10 points)

(b) Prove, that the given scheme is unconditionally stable.

Solution

The scheme is unconditionally stable if

$$|G(k)| \leq 1 \tag{3}$$

See prove in Chapter 1.4.5 in the script with $\theta = 1/2$

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(10 points)

(c) Under what condition the scheme will avoid oscillatory solutions?

Solution

The scheme will avoid oscillatory solutions if

$$G(k) \geq 0 \tag{4}$$

Which is always satisfied for the given scheme (see eq. (1.142) in the script with $\theta = 1/2$)

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(6 points)

(d) For extra 5 points: Prove that the given scheme is convergent.

Solution

As the given scheme is the trapezoidal rule (Theta method with second order error), the proof is in the script (See Chapters 1.4.1 and 1.4.4.)

For 2D Taylor series, refer: `http://www.math.ubc.ca/~feldman/m200/taylor2dSlides.pdf`

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(5 points)

Exercise 3: *Analytical and numerical solution to the heat equation* **(15 points)**

Consider the instationary heat equation

$$\frac{\partial u}{\partial t} - \beta^2 \Delta u = 0$$

on the interval $[0, L]$ with $\beta = 0.5$ and $L = 10$, with the initial condition

$$u(x,0) = \sin(\frac{\pi}{L}x) + 2\sin(3\frac{\pi}{L}x) + \sin(5\frac{\pi}{L}x)$$

and with the boundary conditions

$$u(0,t) = 0$$

$$u(L,t) = 0$$

(a) **Analytical solution**

We know from the lecture that the general solution of the equation with the given boundary conditions is:

$$u(x,t) = \sum_{k=1}^{\infty} B_k e^{-\kappa_k^2 t} \sin(\frac{\kappa_k}{\beta}x)$$

with

$$\kappa_k = \frac{\beta k \pi}{L}$$

Define the coefficients B_k from the initial condition and use a numerical software (Matlab, Python, etc.) to plot the solution at t=0 and T=10.

Solution

Orthogonality of sine functions:

$$\int_0^L \sin(\frac{m\pi x}{L}) \sin(\frac{n\pi x}{L}) dx = \begin{cases} 0, & \text{if } m \neq n \\ \frac{L}{2}, & m = n \end{cases}$$

Given:

$$u(x,t) = \sum_{k=1}^{\infty} B_k e^{-\kappa_k^2 t} \sin(\frac{\kappa_k}{\beta}x)$$

Multiply $\sin(\frac{k\pi x}{L})$ on both sides and at t=0 substitute the given initial condition for LHS and use the above given orthogonality to get:

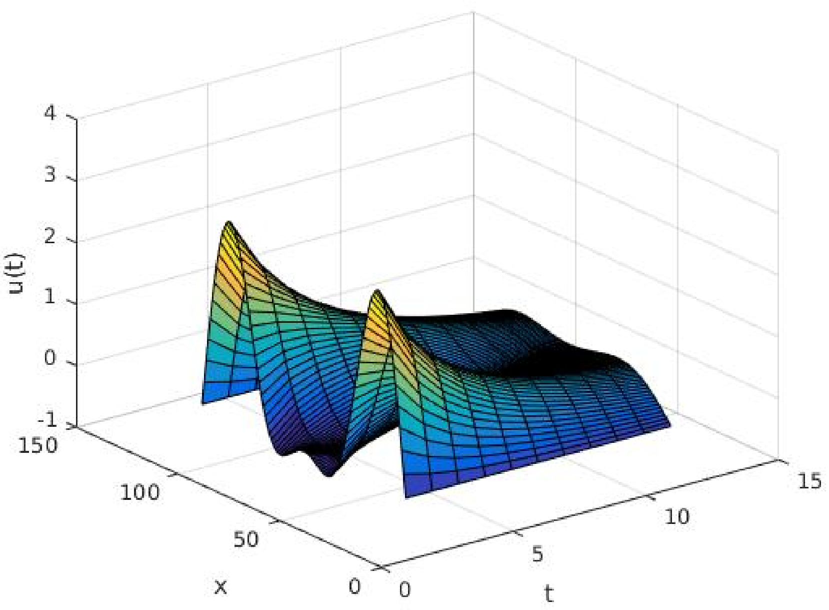
$$B_k = \frac{2}{L} \int_0^L \left\{ \sin(\frac{\pi x}{L}) + 2\sin(\frac{3\pi x}{L}) + \sin(\frac{5\pi x}{L}) \sin \right\} (\frac{k\pi x}{L})$$

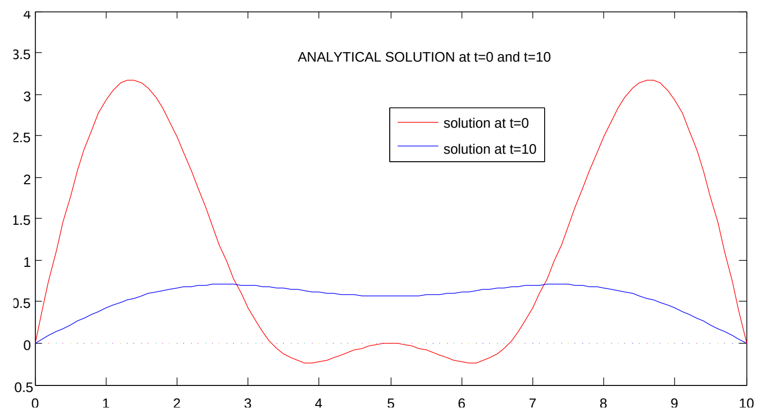
For $k = 1, 3, 5$ we get non-zero RHS and for k $\neq$ { 1,3,5} RHS = 0 (recall Orthogonality of sine functions)

Hence, $B_1 = 1$ $B_3 = 2$ $B_5 = 1$ and rest are 0 and the general solution is:

$$u(x,t) = e^{-\left(\frac{\beta\pi}{L}\right)^2 t} \sin\left(\frac{\pi}{L}x\right) + 2e^{-\left(\frac{3\beta\pi}{L}\right)^2 t} \sin\left(\frac{3\pi}{L}x\right) + e^{-\left(\frac{5\beta\pi}{L}\right)^2 t} \sin\left(\frac{5\pi}{L}x\right)$$

Plotting the solution at t=0 and at t=T=10 gives:





The two bigger peaks at the initial state warm up the part of the rod that is in between the peaks and the temperature is decreasing continuously in the bar. The peaks are almost completely smoothed down at T=10. Physically: Since there is no source term in the PDE, the temperatures should decrease over time. Hence the above obtained behavior corresponds to the physically expected one .

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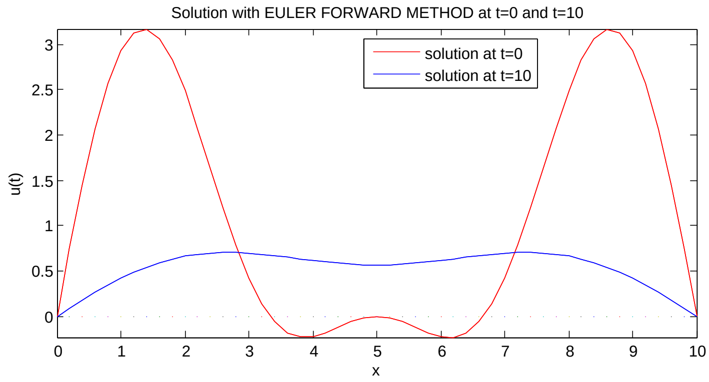
(4 points)

(b) **Solution with Euler forward**

Solve the discretized system with the Euler forward method. Choose $h = 0.2$ for the space discretization and $\Delta t = 0.05$ as time step. Determine the solution at time $T = 10$. Show a plot of the solution at $t = 0$ and at $t = T$.

Solution



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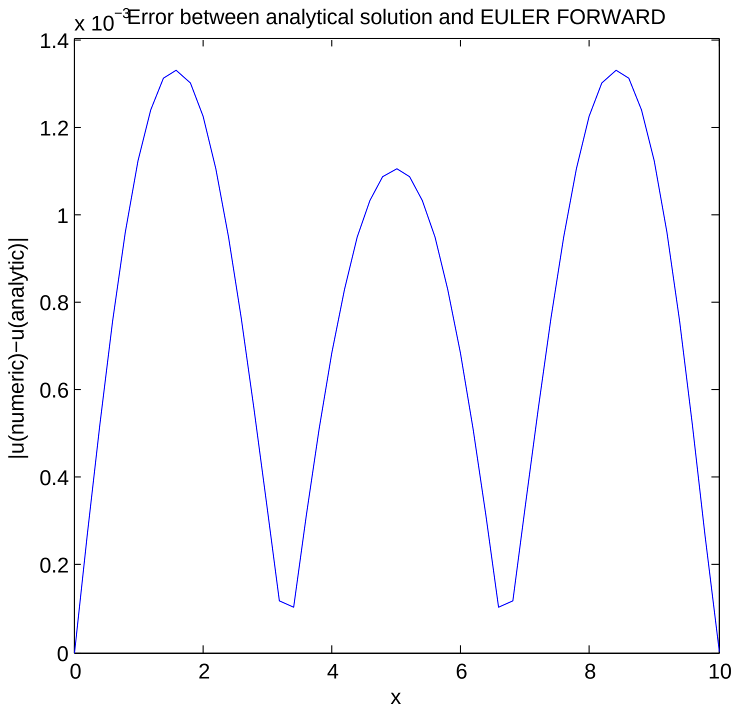
Compare the plots with the analytical solution.

Solution



The errors are of order 10^{-3}

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(6 points)

(c) **Stability (Euler forward)**

Repeatedly double the time step size until the numerical solution becomes unstable. At which Δt

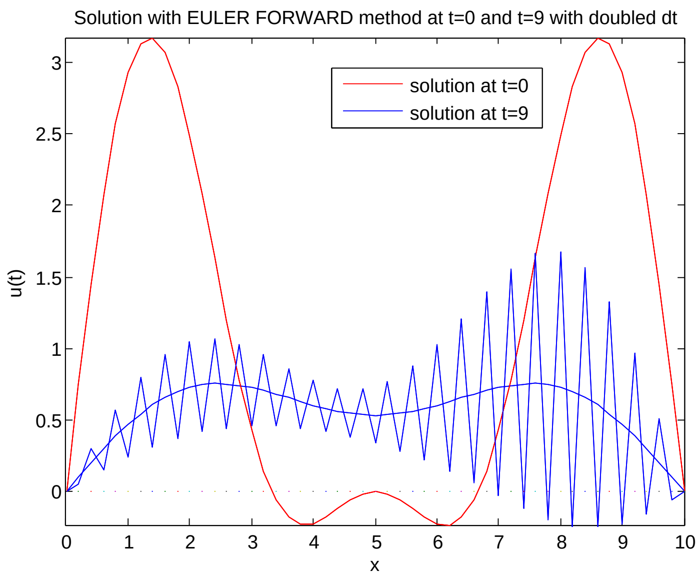
does the solution becomes unstable? Compare to the theoretical value you can deduce from the formula from the lecture (or the script).

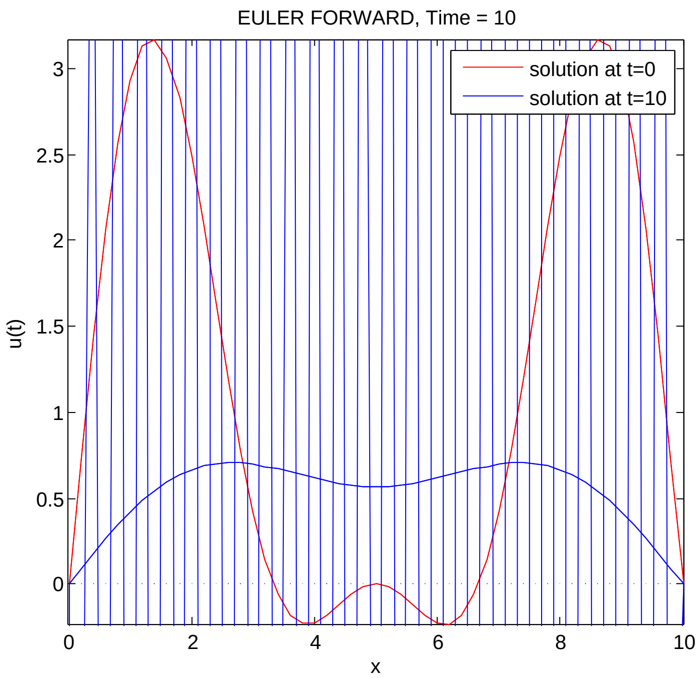
Solution

Doubling the timestep ($\Delta t = 0.1$) turns the Euler forward method instable. As the smallest eigenvalue of the matrix B is not far away smaller than one, it starts only oscilating around t=9. In other words the stability criteria for Euler forward method is not satisfied:

$$\Delta t < \frac{h^2}{2\beta^2} = \frac{(0.2)^2}{2(0.5)^2} = 0.08$$

See a plot of the analytical solution and the one with the Euler forward method at t=9 and 10.





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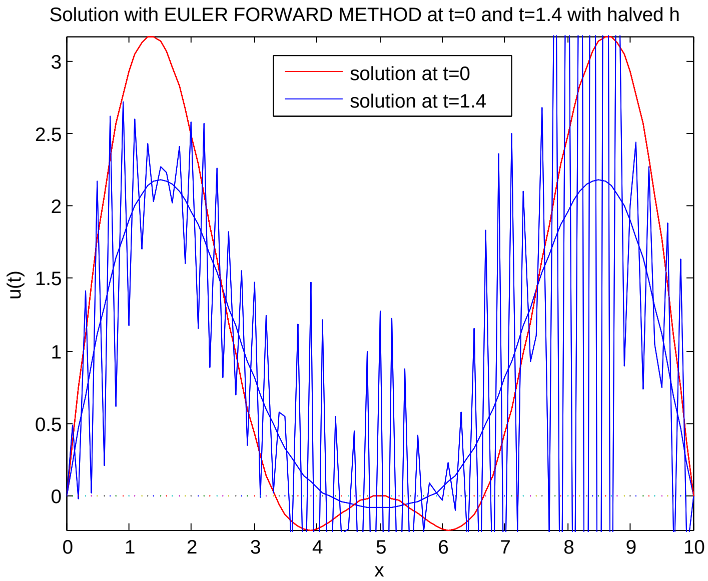
Do the same as above by halving h.

Solution

Halving h ($h = 0.1$) also turns the Euler forward method instable as:

$$h_{cr} = \sqrt{2\Delta t\beta^2} = \sqrt{2(0.05)(0.5)^2} = 0.16 < h$$

It can be seen from the plot bellow that even before t=2 the solution with Euler forward method is already unacceptable.



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(5 points)