Institute of Scientific Computing
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Winter Term 2016
Due time and date:
Monday, 12. December 2016

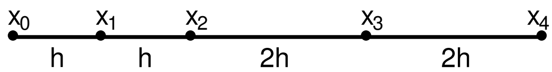
Introduction to PDEs and Numerical Methods (PDEs 1): Assignment 5 (60 points)

Exercise 1: *FDM and Irregular Meshes* **(14 points)**

After a spatial discretisation using the Finite Difference method the instationary heat equation can be expressed as

$$\dot{u}(t) = Au(t) + f$$

where matrix A encapsulates the second order differences. Until now, we have considered a regular mesh, that means, the discrete spatial points were chosen equidistantly. Now, we switch to irregular meshes, which are commonly not used when applying FDM. In this exercise you shall answer the question, why irregular meshes do not seem to be attractive here. Consider the following discrete mesh:



Dirichlet conditions are given by $u_0(t) = u_4(t) = 0$

(a) The formulas for approximating $\frac{\partial^2 u}{\partial x^2}(x_1)$ and $\frac{\partial^2 u}{\partial x^2}(x_3)$ can be taken from the script (e.g. pp 17-18). Use two Taylor expansions around x_2 and sum them up in a weighted manner to derive a formula approximating $\frac{\partial^2 u}{\partial x^2}(x_2)$. What can you say about the order of the error of your formula compared to the one of a regular mesh?

(8 points)

(b) Build up matrix A for the given irregular mesh. What can be noticed compared to the matrix resulting from a regular mesh? (6 points)

Exercise 2: *Neumann stability analysis* **(31 points)**

Apply the von Neumann stability analysis for checking the stability of the following scheme for the instationary heat equation:

$$\frac{u_{j,n+1} - u_{j,n}}{\Delta t} = \frac{\beta^2}{2(\Delta x)^2}\left(u_{j-1,n+1} + u_{j-1,n} - 2u_{j,n+1} - 2u_{j,n} + u_{j+1,n+1} + u_{j+1,n}\right)$$

(a) Determine the gain factor $G(k)$ for the given scheme.

(10 points)

(b) Prove, that the given scheme is unconditionally stable.

(10 points)

(c) Under what condition the scheme will avoid oscillatory solutions?

(6 points)

(d) For extra 5 points: Prove that the given scheme is convergent.

(5 points)

Exercise 3: *Analytical and numerical solution to the heat equation* **(15 points)**

Consider the instationary heat equation

$$\frac{\partial u}{\partial t} - \beta^2 \Delta u = 0$$

on the interval $[0, L]$ with $\beta = 0.5$ and $L = 10$, with the initial condition

$$u(x,0) = \sin(\frac{\pi}{L}x) + 2\sin(3\frac{\pi}{L}x) + \sin(5\frac{\pi}{L}x)$$

and with the boundary conditions

$$u(0,t) = 0$$

$$u(L,t) = 0$$

(a) **Analytical solution**

We know from the lecture that the general solution of the equation with the given boundary conditions is:

$$u(x,t) = \sum_{k=1}^{\infty} B_k e^{-\kappa_k^2 t} \sin(\frac{\kappa_k}{\beta}x)$$

with

$$\kappa_k = \frac{\beta k \pi}{L}$$

Define the coefficients B_k from the initial condition and use a numerical software (Matlab, Python, etc.) to plot the solution at t=0 and T=10.

(4 points)

(b) **Solution with Euler forward**

Solve the discretized system with the Euler forward method. Choose $h = 0.2$ for the space discretization and $\Delta t = 0.05$ as time step. Determine the solution at time $T = 10$. Show a plot of the solution at $t = 0$ and at $t = T$. Compare the plots with the analytical solution. (6 points)

(c) **Stability (Euler forward)**

Repeatedly double the time step size until the numerical solution becomes unstable. At which Δt does the solution becomes unstable? Compare to the theoretical value you can deduce from the formula from the lecture (or the script). Do the same as above by halving h. (5 points)