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Winter Term 2016
Due time and date:
9:00 o'clock, 05. Dec. 2015

Introduction to PDEs and Numerical Methods (PDEs 1): Assignment 4 (50 points)

Exercise 1: *Symmetric operator* **(6 points)**

Consider a function space with mixed boundary conditions

$$V = \{u \in C^2((0,l)) : u(0) = 0 \text{ and } \frac{du}{dx}(l) = 0\} \tag{1}$$

and a differential operator $L_M : V \to V$ defined as

$$L_M u(x) = -\frac{d^2}{dx^2} u(x).$$

Show that the operator is a symmetric (self-adjoint) one with respect to the following inner product (scalar product) $(v, w) = \int_0^l v(x)w(x)dx$.

Solution

If the operator is symmetric $(L_M u(x), v(x)) = (u(x), L_M v(x))$, where $u(x), v(x) \in V$. Therefore, $v(0) = u(0) = 0$, and $\frac{du}{dx}(l) = \frac{dv}{dx}(l) = 0$

$$(L_M u(x), v(x)) = -\int_0^l \frac{d^2u(x)}{dx^2} v(x)dx = -\frac{du(x)}{dx} v(x)\Big|_0^l + \int_0^l \frac{du(x)}{dx}\frac{dv(x)}{dx}dx$$

$$= -\left(\underbrace{\frac{du(l)}{dx}}_{0} v(l) - \frac{du(0)}{dx}\underbrace{v(0)}_{0}\right) + \int_0^l \frac{du(x)}{dx}\frac{dv(x)}{dx}dx$$

$$= u(x)\frac{dv(x)}{dx}\Big|_0^l - -\int_0^l u(x)\frac{d^2v(x)}{dx^2}dx$$

$$= \left(u(l)\underbrace{\frac{dv(l)}{dx}}_{0} - \underbrace{u(0)}_{0}\frac{dv(0)}{dx}\right) - \int_0^l u(x)\frac{d^2v(x)}{dx^2}dx = (u(x), L_M v(x))$$

Thus, the operator is symmetric.

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Exercise 2: *Eigenvalues, eigenvectors* **(12 points)**

Let $A \in \mathbb{R}^{nxn}$ be a symmetric positive definite matrix and $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be the eigenvalues of A.

(a) Show that A^{-1} has eigenvalues $1/\lambda_i, i = 1..n$ and show also, that the eigenvectors of A are also eigenvectors of A^{-1}. (6 points)

Solution

Let v_i be the eigenvector corresponding to the eigenvalue λ_i. Then

$$Av_i = \lambda_i v_i$$

Multiplying by A^{-1}, then

$$A^{-1}Av_i = A^{-1}\lambda_i v_i$$

Since λ_i is an scalar and $A^{-1}A = I$, where I is the identity, we get

$$v_i = \lambda_i A^{-1} v_i$$

$$\frac{1}{\lambda_i} v_i = A^{-1} v_i$$

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(b) Which will be the largest and which the smallest eigenvalue of A^{-1}? (6 points)

Solution

Since the eigenvalues of $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ of A, then the eigenvalues of A^{-1} are $\frac{1}{\lambda_1} \geq \frac{1}{\lambda_2} \geq \cdots \geq \frac{1}{\lambda_n} > 0$. Therefore, the biggest eingenvalue of A^{-1} is $\frac{1}{\lambda_1}$ and the smallest one is $\frac{1}{\lambda_n}$

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Exercise 3: *FD approximation of the Poisson equation with mixed B.C.s* **(32 points)**

Consider the boundary value problem

$$-u''(x) = f(x) \quad u(0) = 0 \quad u'(1) = 1,$$

This problem has a Dirichlet boundary condition at $x = 0$ and a Neumann boundary condition at $x = 1$, which can be discretised by the usual difference formula

$$u'_n = \frac{u_n - u_{n-1}}{h} = 1.$$

(a) Show that this approximation of the Neumann condition is of order 1.

Solution

$$u(x-h) = u(x) - u'(x)h + \frac{1}{2}u''(x)h^2 + O(h^3)$$

$$u'(x) = \frac{u(x) - u(x-h)}{h} + \frac{1}{2}u''(x)h + O(h^2) = \frac{u(x) - u(x-h)}{h} + O(h)$$

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(3 points)

(b) Write down the discretisation of this problem, if central differences are used. Use a matrix-vector notation!
Is the system matrix symmetric?

Solution

$$\frac{1}{h^2}\begin{bmatrix} 2 & -1 & & & & 0 \\ -1 & 2 & -1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & & -1 & 2 & -1 \\ 0 & & & & -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-1} \\ u_n \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_{n-1} \\ \frac{1}{h} \end{bmatrix}$$

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(6 points)

(c) Let $f(x) = -e^{x-1}$. Show analytically that

$$u(x) = e^{-1}(e^x - 1)$$

is a solution to our problem.

Solution

First let's check whether the boundary conditions are satisfied:

$$u(0) = e^{-1}(e^0 - 1) = 0 \to \checkmark$$

$$u'(x) = e^{-1}e^x$$

$$u'(1) = e^{-1}e^1 = 1 \to \checkmark$$

Now let's check if the Poisson equation is satisfied:

$$u''(x) = e^{-1}e^x = e^{x-1} \to \checkmark$$

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(5 points)

(d) Write a MATLAB program which solves the problem numerically. Use the discretisation from part b). Solve the problem for different stepsizes h. Give the order of the approximation from your MATLAB code.

Solution

The central difference approximation of the second derivative is a second order approximation (see sketch of Tutorial 3), but because of the approximation of the Neumann B.C., it is of order 1.
From MATLAB, we find the order of approximation to be 0.98

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```
clc
close all
clear all

deltax = [0.25 0.2 0.1 0.05 0.01];
uAna = @(x) exp(-1)*(exp(x)-1);

fplot(uAna,[0,1],'Linewidth',2);
hold on

for k = 1:length(deltax)

h = deltax(k);
n = 1/h;
x = [0:h:1];
A = zeros(n,n);

u = zeros(n,1);



for i=1:n
f(i) = -(exp(x(i+1)-1)); % Ignoring 0th node,considering the last node
for j = 1:n

if(i==j)
    A(i,j) = 2/(h^2);
end
if(i==j+1 || j==i+1)
    A(i,j) = -1/(h^2);
end

end


end
A(n,n) = 1/(h^2);
f(n) = 1/h;
%Solving the system Au = f
u = (A\f');

NSol = zeros(n+1,1);
for i =2:n+1
NSol(i)= u(i-1);
end
coulourstring = 'kbgyr';
plot(x,NSol,'Linewidth',2,'Color',coulourstring(k));
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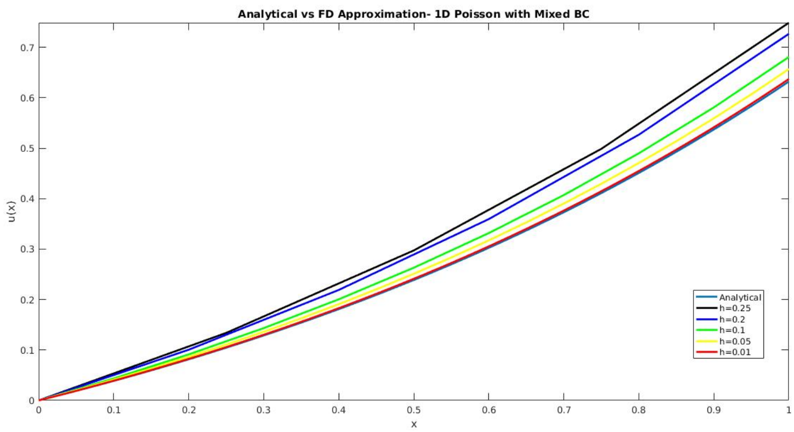
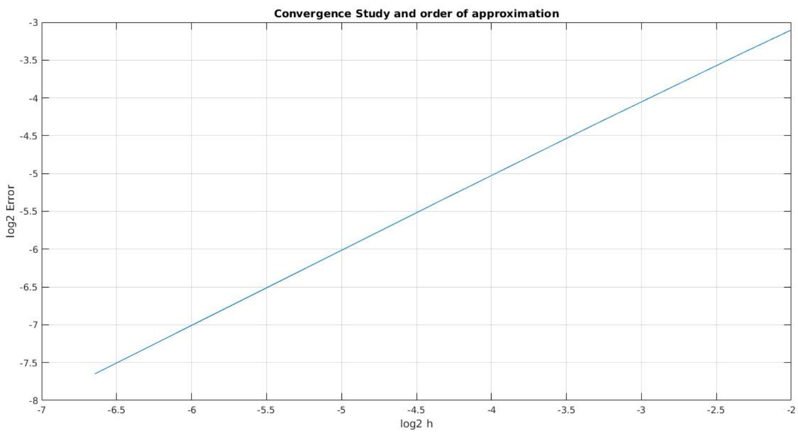
```
hold on
UAna = exp(-1)*(exp(x)-1);
error(k) = norm((NSol-UAna'),inf);

end
% Analytical Solution
hold off
xlabel('x')
ylabel('u(x)');
title('Analytical vs FD Approximation- 1D Poisson with Mixed BC')
% legend('Numerical Approximation','Analytical')
legend('Analytical','h=0.25','h=0.2','h=0.1','h=0.05','h=0.01')

figure
plot(log2(deltax),log2(error));
xlabel('log2 h')
ylabel('log2 Error')
title('Convergence Study and order of approximation')
grid on
graph = polyfit(log2(deltax),log2(error),1);
slope = graph(1);
fprintf('The order of the Approximation is : %d \n',slope);
```





(10 points)

(e) Next we approximate the Neumann boundary condition with

$$\frac{3u_n - 4u_{n-1} + u_{n-2}}{2h} = 1.$$

Show that this approximation is of order 2, write down the discretisation as in b), modify your MATLAB code and check the numerical convergence order.

Solution

$$\begin{aligned}
4u_{n-1} = 4u(x-h) &= 4u(x) - 4u'(x)h + \frac{4}{2}u''(x)h^2 - \frac{4}{3!}u'''(x)h^3 + O(h^4) \\
u_{n-2} = u(x-2h) &= u(x) - 2u'(x)h + \frac{4}{2}u''(x)h^2 - \frac{8}{3!}u'''(x)h^3 + O(h^4) \\
\frac{-4u_{n-1} + u_{n-2}}{2h} &= \frac{-3}{2h}u(x) + u'(x) + 0 + O(h^2) \\
u'(x) &= \frac{3u_n - 4u_{n-1} + u_{n-2}}{2h} + O(h^2)
\end{aligned}$$

The convergence order is 2.

$$\frac{1}{h^2}\begin{bmatrix} 2 & -1 & & & & 0 \\ -1 & 2 & -1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & & -1 & 2 & -1 \\ 0 & & & 0.5 & -2 & 1.5 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-1} \\ u_n \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_{n-1} \\ \frac{1}{h} \end{bmatrix}$$

The MATLAB code from 3(d) can be slightly modified as per the above given matrix. We obtain an order of approximation = 1.93 indicating it to be a second order convergence.

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(8 points)