Institute of Scientific Computing
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Winter Term 2016
Due time and date:
9:00 o'clock, 05. Dec. 2015

Introduction to PDEs and Numerical Methods (PDEs 1): Assignment 4 (50 points)

Exercise 1: *Symmetric operator* **(6 points)**

Consider a function space with mixed boundary conditions

$$V = \{u \in C^2((0,l)) : u(0) = 0 \text{ and } \frac{du}{dx}(l) = 0\} \tag{1}$$

and a differential operator $L_M : V \to V$ defined as

$$L_M u(x) = -\frac{d^2}{dx^2}u(x).$$

Show that the operator is a symmetric (self-adjoint) one with respect to the following inner product (scalar product) $(v, w) = \int_0^l v(x)w(x)dx$.

Exercise 2: *Eigenvalues, eigenvectors* **(12 points)**

Let $A \in \mathbb{R}^{nxn}$ be a symmetric positive definite matrix and $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$ be the eigenvalues of A.

(a) Show that A^{-1} has eigenvalues $1/\lambda_i, i = 1..n$ and show also, that the eigenvectors of A are also eigenvectors of A^{-1}. (6 points)

(b) Which will be the largest and which the smallest eigenvalue of A^{-1}? (6 points)

Exercise 3: *FD approximation of the Poisson equation with mixed B.C.s* **(32 points)**

Consider the boundary value problem

$$-u''(x) = f(x) \quad u(0) = 0 \quad u'(1) = 1,$$

This problem has a Dirichlet boundary condition at $x = 0$ and a Neumann boundary condition at $x = 1$, which can be discretised by the usual difference formula

$$u_n' = \frac{u_n - u_{n-1}}{h} = 1.$$

(a) Show that this approximation of the Neumann condition is of order 1. (3 points)

(b) Write down the discretisation of this problem, if central differences are used. Use a matrix-vector notation!

Is the system matrix symmetric? (6 points)

(c) Let $f(x) = -e^{x-1}$. Show analytically that

$$u(x) = e^{-1}(e^x - 1)$$

is a solution to our problem.

(5 points)

(d) Write a MATLAB program which solves the problem numerically. Use the discretisation from part b). Solve the problem for different stepsizes h. Give the order of the approximation from your MATLAB code.

(10 points)

(e) Next we approximate the Neumann boundary condition with

$$\frac{3u_n - 4u_{n-1} + u_{n-2}}{2h} = 1.$$

Show that this approximation is of order 2, write down the discretisation as in b), modify your MATLAB code and check the numerical convergence order. (8 points)