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Introduction to PDEs and Numerical Methods (PDEs 1): Assignment 2 (50 points)

Exercise 1: *Subspaces, orthogonal projection* **(16 points)**

Define S to be the set of all polynomials of the form $ax+bx^2$, considered as functions defined on the interval $[0,1]$.

(a) Explain why S is a subspace of $C^2[0,1]$ (4 points)

(b) Compute the approximation from S, to $f(x) = e^x$ by minimizing the induced norm

$$||u|| = \sqrt{\langle u, u\rangle}$$

using the inner product:

$$a(u,v) = \langle u, v\rangle = \int_0^1 u(x)v(x)dx,$$

and plot the original function and its approximation with a suitable software (e.g. MATLAB, PYTHON).

Solution

(a)

The first derrivative:

$$\frac{\partial(ax+bx^2)}{\partial x} = a + 2bx \tag{1}$$

The second derrivation:

$$\frac{\partial(a+2bx)}{\partial x} = 2b \tag{2}$$

Since the first and second derrivative exist and are continuous in the given interval [0,1], S is a subspace of C^2 - the space containing functions which are two times differentiable.

(b)

The given space S is: S = span$\{x$,$x^2\}$ = span$\{\phi_i\}$

Say, $f_h = \sum_{i=1}^2 \alpha_i\phi_i$ is the best approximation of e^x from S,

where

ϕ_1 = x and $\phi_2 = x^2$

We know that the error due to approximation will be minimum if the error $f - f_h$ under the given norm is orthogonal to the given space S. This means that the error is perpendicular to each of the basis functions which span the given space S. Mathematically,

error = f - $f_h = e^x - \sum_{i=1}^2 \alpha_i\phi_i$

<f - f_h, ϕ_j> = 0 j = 1,2

Therefore we have,

$\sum_{i=1}^2 \alpha_i < \phi_i, \phi_j > = < f, \phi_j >$ j = 1,2

Using the given inner product, the following matrix is obtained:

$$\begin{bmatrix} \int_0^1 \phi_1\phi_1 dx & \int_0^1 \phi_1\phi_2 dx \\ \int_0^1 \phi_2\phi_1 dx & \int_0^1 \phi_2\phi_2 dx \end{bmatrix} * \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} \int_0^1 f(x)\phi_1 dx \\ \int_0^1 f(x)\phi_2 dx \end{bmatrix}$$

$$\begin{bmatrix} \int_0^1 x * x dx & \int_0^1 x * x^2 dx \\ \int_0^1 x^2 * x dx & \int_0^1 x^2 * x^2 dx \end{bmatrix} * \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} \int_0^1 e^x * x \\ \int_0^1 e^x * x^2 dx \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{3} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{5} \end{bmatrix} * \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 1 \\ e-2 \end{bmatrix}$$

Solving the above system, we get α_1 = 4.90314 and α_2 = -2.53752

Hence we have the best approximation for e^x from S using the above given norm is $f_h = 4.90314x - 2.53752x^2$

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% PDE 1-Assignemnet 2-Excercise 1(b) - Solution
% Given function in the interval [0 1]
fplot(@(x)exp(x),[0 1])
hold on
% Best approximation obtained from the space S is f_h = 4.90314x - 2.53752x^2
f = @(x) 4.90314*x - 2.53752*x^2;
fplot(f,[0 1],'r')
title('Best approximation of e^x from the space of second degree polynomials')
xlabel('x')
ylabel('f_h(x)')
legend('f(x)', 'Best approximation f_h(x)')
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(12 points)

Exercise 2: *Fourier Series* **(12 points)**

Determine the Fourier series of the function $f[-1,1] \to \mathbb{R}$ with $f(x) := (\pi x)^2$. Plot the original function and the first 5 Fourier terms with a suitable software (e.g. MATLAB or PYTHON).

Solution

$$
\begin{aligned}
a_0 &= \int_{-1}^{1} (\pi x)^2 dx = \left[\frac{\pi^2 x^3}{3}\right]_{-1}^{1} = \frac{2\pi^2}{3} \\
a_k &= \int_{-1}^{1} (\pi x)^2 \cos(k\pi x) dx \\
&= \pi^2 \int_{-1}^{1} \underbrace{x^2}_{u} \underbrace{\cos(k\pi x)}_{v'} dx \\
&= \frac{\pi}{k}\left[x^2 \sin(k\pi x)\right]_{-1}^{1} - \frac{\pi}{k}\int_{-1}^{1} 2 \underbrace{x}_{u} \underbrace{\sin(k\pi x)}_{v'} dx \\
&= 0 + \frac{2}{k^2}\left[x\cos(k\pi x)\right]_{-1}^{1} - \frac{2}{k^2}\underbrace{\int_{-1}^{1} \cos(k\pi x) dx}_{=0} \\
&= \frac{2}{k^2}\left(\cos(k\pi) + \cos(-k\pi)\right) \\
&= \begin{cases} \frac{-4}{k^2} & k \text{ odd} \\ \frac{4}{k^2}, & \text{even} \end{cases}. \\
b_k &= \int_{-1}^{1} (\pi x)^2 \sin(k\pi x) dx = 0
\end{aligned}
$$

Accordingly the expansion is:

$$\hat{f}(x) = \frac{\pi^2}{3} + 4\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \cos k\pi x.$$

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% PDE 1-Assignemnet 2-Excercise 2 - Solution

% Given function in the interval [-1 1]
figure
fplot(@(x)pi*pi*x*x,[-1 1])
hold on
% Fourier expansion of the function up to 5th term.
f = @(x) (pi*pi/3) - 4*cos(pi*x) + cos(2*pi*x)-(4/9)*cos(3*pi*x)+ (1/4)*cos(4*pi*x);
fplot(f,[-1 1],'r')
title('Fourier repesentation of a Function')
xlabel('x')
ylabel('f(x)')
legend('f(x)', 'Fourier Representation of f(x)')
%                           %******************OBSERVATION FROM THE PLOT*****************
% There is close agreement of the given function with its fourier representation.
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Exercise 3: *Norms and inner products* **(10 points)**

(a) Consider the vectors:

$$\mathbf{v}_1 = \begin{bmatrix} -9 \\ 16 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Compute the following expressions:

- $||\mathbf{v}_1||_1$, $||\mathbf{v}_2||_1$
- $||\mathbf{v}_1||_2$, $||\mathbf{v}_2||_2$
- $||\mathbf{v}_1||_\infty$, $||\mathbf{v}_1||_\infty$
- $||\mathbf{v}_1||_4$, $||\mathbf{v}_2||_4$
- $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle$

(5 points)

Solution

Definition of norms:
$||x||_1 = \sum_{i=1}^{n} |x_i|$
$||x||_2 = (\sum_{i=1}^{n} |x_i|^2)^(\frac{1}{2}$
$||x||_\infty = max_{i\epsilon N}|x_i|$
$||x||_4 = (\sum_{i=1}^{n} |x_i|^4)^(\frac{1}{4}$
$\langle x, y \rangle = \sum_{i=1}^{n} x_i * y_i$

Using the above, we get the following results:
$||v_1||_1 = |-9| + 16 = 25$
$||v_2||_1 = 1 + |-1| = 2$
$||v_1||_2 = \sqrt[2]{(-9)^2 + 16^2} = 18.3575$
$||v_2||_2 = \sqrt[2]{1^2 + (-1)^2} = 1.4142$
$||v_1||_\infty = 16$
$||v_2||_\infty = 1$
$||v_1||_4 = \sqrt[4]{(-9)^4 + 16^4} = 16.3862$
$||v_2||_4 = \sqrt[4]{1^4 + (-1)^4} = 1.1892$
$\langle x, y \rangle = (-9 * 1) + (16 * -1) = -25$

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(b) Consider the scalar functions:

$$f(x) = \cos(\pi x), \quad g(x) = 2 \quad x \in \Omega = [-1, 1]$$

Compute the following expressions on the domain Ω:

- $||f||_2$, $||g||_2$
- $||f||_\infty$, $||g||_\infty$
- $\langle f, g \rangle$

(5 points)

Solution

- $||f||_2 = \sqrt[2]{\int_{-1}^{1} |f(x)|^2) dx} = \sqrt[2]{\int_{-1}^{1} cos^2(\pi x)} dx = 1$
- $||g||_2 = \sqrt[2]{\int_{-1}^{1} |g(x)|^2) dx} = \sqrt[2]{\int_{-1}^{1} 2^2 dx} = \sqrt[2]{8} = 2.8284$
- $||f||_\infty = sup_{x \epsilon \Omega} |f(x)| = |\cos(-\pi)| = |\cos(\pi)| = 1$
- $||g||_\infty = 2$
- $\langle f, g \rangle = \int_{-1}^{1} f(x) g(x) dx = \int_{-1}^{1} 2 cos(\pi x) dx = 0$

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Exercise 4: *Inner product* **(10 points)**

Prove that if $\mathbf{A} \in \mathbb{R}^{n \times n}$ is a symmetric and positive definite matrix, that is

$$\mathbf{x}^T \mathbf{A} \mathbf{x} \geq 0 \quad \forall \mathbf{x} \in \mathbb{R}^{\mathbf{n}} \quad \text{and} \quad \mathbf{x}^T \mathbf{A} \mathbf{x} = 0 \quad \text{onlywhen} \quad \mathbf{x} = \mathbf{0}$$

then the mapping

$$f(\mathbf{x}, \mathbf{y}) = \mathbf{x}^T \mathbf{A} \mathbf{y}$$

defines an inner product.

Solution

Symmetry:

$$f(\mathbf{x}, \mathbf{y}) = f(\mathbf{y}, \mathbf{x})$$

As the mapping $f(\mathbf{x}, \mathbf{y})$ gives a scalar, it can be transposed without changing it:

$$f(\mathbf{x}, \mathbf{y}) = \mathbf{x}^T \mathbf{A} \mathbf{y} = \left(\mathbf{x}^T \mathbf{A} \mathbf{y}\right)^T = \mathbf{y}^T \mathbf{A}^T \left(\mathbf{x}^T\right)^T = \mathbf{y}^T \mathbf{A} \mathbf{x} = f(\mathbf{y}, \mathbf{x})$$

Where the last but one equation holds due to the symmetry of $\mathbf{A}$: $\mathbf{A}^T = \mathbf{A}$

Linearity in the first argument:

$$f(\alpha \mathbf{x}, \mathbf{y}) = \alpha f(\mathbf{x}, \mathbf{y})$$

$$f(\mathbf{x} + \mathbf{y}, \mathbf{z}) = f(\mathbf{x}, \mathbf{z}) + f(\mathbf{y}, \mathbf{z})$$

$$f(\alpha \mathbf{x}, \mathbf{y}) = (\alpha \mathbf{x})^T \mathbf{A} \mathbf{y} = \alpha (\mathbf{x}^T \mathbf{A} \mathbf{y}) = \alpha f(\mathbf{x}, \mathbf{y})$$

$$f(\mathbf{x} + \mathbf{y}, \mathbf{z}) = (\mathbf{x} + \mathbf{y})^T \mathbf{A} \mathbf{z} = \left(\mathbf{x}^T + \mathbf{y}^T\right) \mathbf{A} \mathbf{z} = \mathbf{x}^T \mathbf{A} \mathbf{z} + \mathbf{y}^T \mathbf{A} \mathbf{z} = f(\mathbf{x}, \mathbf{z}) + f(\mathbf{y}, \mathbf{z})$$

Positive-definiteness:

$$f(\mathbf{x}, \mathbf{x}) \geq 0$$

$$f(\mathbf{x}, \mathbf{x}) = 0 \Rightarrow \mathbf{x} = \mathbf{0}$$

$$f(\mathbf{x},\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x} \geq 0$$

$$f(\mathbf{x},\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x} = 0 \Rightarrow \mathbf{x} = \mathbf{0}$$

Because of the positive-definiteness of the matrix $\mathbf{A}$

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Exercise 5: *PDE and Boundary Conditions* **(2 points)**

(a) For what values of α, is the PDE hyperbolic?

$$\frac{\partial^2 u}{\partial t^2} - \alpha\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial u}{\partial x} = 0$$

(2 points)

Solution

General form : $Au_{xx} + Bu_{xy} + Cu_{xy}$ + lower order terms $= 0$

To be hyperbolic, $AC - B^2 < 0$ and here, $A = -\alpha, B = 0, C = 1$

We get $-\alpha < 0$. Hence for any positive value of α the given PDE is hyperbolic.

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