**Institute of Scientific Computing**
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# Introduction to PDEs and Numerical Methods (PDEs 1): Assignment 2 (50 points)

**Exercise 1:** *Subspaces, orthogonal projection* **(16 points)**

Define $S$ to be the set of all polynomials of the form $ax + bx^2$, considered as functions defined on the interval $[0, 1]$.

(a) Explain why $S$ is a subspace of $C^2[0, 1]$ (4 points)

(b) Compute the approximation from $S$, to $f(x) = e^x$ by minimizing the induced norm

$$||u|| = \sqrt{\langle u, u\rangle}$$

using the inner product:

$$a(u, v) = \langle u, v\rangle = \int_0^1 u(x)v(x)dx,$$

and plot the original function and its approximation with a suitable software (e.g. MATLAB, PYTHON). (12 points)

**Exercise 2:** *Fourier Series* **(12 points)**

Determine the Fourier series of the function $f[-1, 1] \to \mathbb{R}$ with $f(x) := (\pi x)^2$. Plot the original function and the first 5 Fourier terms with a suitable software (e.g. MATLAB or PYTHON).

**Exercise 3:** *Norms and inner products* **(10 points)**

(a) Consider the vectors:

$$\mathbf{v}_1 = \begin{bmatrix} -9 \\ 16 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Compute the following expressions:

- $||\mathbf{v}_1||_1$, $||\mathbf{v}_2||_1$
- $||\mathbf{v}_1||_2$, $||\mathbf{v}_2||_2$
- $||\mathbf{v}_1||_\infty$, $||\mathbf{v}_1||_\infty$
- $||\mathbf{v}_1||_4$, $||\mathbf{v}_2||_4$
- $\langle \mathbf{v}_1, \mathbf{v}_2\rangle$

(5 points)

(b) Consider the scalar functions:

$$f(x) = \cos(\pi x), \quad g(x) = 2 \quad x \in \Omega = [-1, 1]$$

Compute the following expressions on the domain $\Omega$:

- $||f||_2$, $||g||_2$
- $||f||_\infty$, $||g||_\infty$
- $\langle f, g \rangle$

(5 points)

**Exercise 4:** *Inner product* **(10 points)**

Prove that if $\mathbf{A} \in \mathbb{R}^{n\times n}$ is a symmetric and positive definite matrix, that is

$$\mathbf{x}^T\mathbf{A}\mathbf{x} \geq 0 \quad \forall \mathbf{x} \in \mathbb{R}^{\mathbf{n}} \quad \text{and} \quad \mathbf{x}^T\mathbf{A}\mathbf{x} = 0 \quad \text{onlywhen} \quad \mathbf{x} = \mathbf{0}$$

then the mapping

$$f(\mathbf{x}, \mathbf{y}) = \mathbf{x}^T\mathbf{A}\mathbf{y}$$

defines an inner product.

**Exercise 5:** *PDE and Boundary Conditions* **(2 points)**

(a) For what values of $\alpha$, is the PDE hyperbolic?

$$\frac{\partial^2 u}{\partial t^2} - \alpha\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial u}{\partial x} = 0$$

(2 points)