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Introduction to PDEs and Numerical Methods (PDEs 1): Assignment 1 (50 points)

Exercise 1: *Differential operators* **(15 points)**

(a) Let $f_1(x, y, z) = ze^x \sin(y)$. Determine $\frac{\partial f_1}{\partial x}$, $\frac{\partial f_1}{\partial y}$, $\frac{\partial f_1}{\partial z}$ and ∇f_1. (4 points)

Solution (a)

$$\frac{\partial f_1}{\partial x} = ze^x \sin(y) \quad \frac{\partial f_1}{\partial y} = ze^x \cos(y) \quad \frac{\partial f_1}{\partial z} = e^x \sin(y) \quad \nabla f_1(x, y) = \begin{bmatrix} ze^x \sin(y) \\ ze^x \cos(y) \\ e^x \sin(y) \end{bmatrix}$$

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(b) Let $\mathbf{f_2}(x, y, z) = (xy^2, xy, \cos(z))^T$. Determine $\nabla \cdot \mathbf{f_2}$ and $\nabla \times \mathbf{f_2}$. (4 points)

Solution

$$\nabla \cdot \mathbf{f_2} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix}^T \begin{bmatrix} xy^2 \\ xy \\ \cos(z)) \end{bmatrix} = \frac{\partial(xy^2)}{\partial x} + \frac{\partial(xy)}{\partial y} + \frac{\partial \cos(z)}{\partial z} = y^2 + x - \sin(z)$$

$$\nabla \times \mathbf{f_2} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \times \begin{bmatrix} xy^2 \\ xy \\ \cos(z)) \end{bmatrix} = \begin{bmatrix} \frac{\partial \cos(z)}{\partial y} - \frac{\partial xy}{\partial z} \\ -\frac{\partial \cos(z)}{\partial x} + \frac{\partial xy^2}{\partial z} \\ \frac{\partial xy}{\partial x} - \frac{\partial xy^2}{\partial y} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ y - 2yx \end{bmatrix}$$

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(c) Let $f_3(x, y, z) = x^2 + y^4 z$. Determine Δf_3. (3 points)

Solution

$\Delta f_3 = \frac{\partial^2 f_3}{\partial x^2} + \frac{\partial^2 f_3}{\partial y^2} + \frac{\partial^2 f_3}{\partial z^2}$.

$\frac{\partial f_3}{\partial x} = 2x$, then $\frac{\partial^2 f_3}{\partial x^2} = 2$

$\frac{\partial f_3}{\partial y} = 4y^3 z$, then $\frac{\partial^2 f_3}{\partial y^2} = 12y^2 z$

$\frac{\partial f_3}{\partial z} = y^4$, then $\frac{\partial^2 f_3}{\partial z^2} = 0$

Therefore, $\Delta f_3 = 2 + 12y^2 z$.

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(d) Show that $\nabla \cdot \nabla f = \Delta f$ and $\nabla \times \nabla f = 0$ for any two-times differentiable function $f : \Omega \to \mathbb{R}^3$. (4 points)

Solution

$\nabla \cdot \nabla f = \frac{\partial}{\partial x}\frac{\partial f}{\partial x} + \frac{\partial}{\partial y}\frac{\partial f}{\partial y} + \frac{\partial}{\partial z}\frac{\partial f}{\partial z} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \Delta f$

Using Matrix-vector Notation:

$\nabla \times \nabla f = \frac{\partial}{\partial x_i}\boldsymbol{e_i} \times \frac{\partial f}{\partial x_j}\boldsymbol{e_j} = \frac{\partial^2 f}{\partial x_i x_j}\epsilon_{ijk}\boldsymbol{e_k} = (\frac{\partial^2 f}{\partial x_2 x_3} - \frac{\partial^2 f}{\partial x_3 x_2})\boldsymbol{e_1} + (\frac{\partial^2 f}{\partial x_3 x_1} - \frac{\partial^2 f}{\partial x_1 x_3})\boldsymbol{e_2} + (\frac{\partial^2 f}{\partial x_1 x_2} - \frac{\partial^2 f}{\partial x_2 x_1})\boldsymbol{e_3}$

Since $\frac{\partial^2 f}{\partial x_i x_j} = \frac{\partial^2 f}{\partial x_j x_1}$, the sum for every one of the vector components is zero.

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Exercise 2: *Heat equation* **(6 points)**

Consider the heat equation on a bar of unit length, with parameter β^2:

$$\frac{\partial}{\partial t}u(x,t) - \beta^2 \frac{\partial^2}{\partial x^2}u(x,t) = f(x,t)$$

(a) Assume boundary conditions $u(0,t) = 0$, $u(\pi,t) = 0$ and the source term $f(x,t) = 0$. Prove that $u(x,t) = e^{-2t}\sin(x)$ can be a solution of the heat equation and specify the value of β^2 that ensures this proof. (3 points)

Solution

At $x = 0, u(0,t) = e^{-2t}sin(0) = 0$ $\quad x = \pi, u(\pi,t) = e^{-2t}sin(\pi) = 0$

Hence the boundary conditions are satisfed!

$\frac{\partial}{\partial t}u(x,t) = -2e^{-2t}\sin(x)$ and $\frac{\partial^2}{\partial x^2}u(x,t) = -e^{-2t}\sin(x)$

Substituting these in the above heat equation, we get:

$-2e^{-2t}\sin(x) + \beta^2 e^{-2t}\sin(x) = 0$

$e^{-2t}\sin(x)[-2 + \beta^2] = 0$

$\beta = \pm\sqrt{2}$

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(b) Now assume $\beta = 1$, boundary conditions $\frac{\partial u}{\partial x}(0,t) = 0$ and $\frac{\partial u}{\partial x}(\pi,t) = 0$ and a solution $u(x,t) = (t^2+t)\cos(x)$. What must $f(x,t)$ look like if the heat equation should be satisfied.

(3 points)

Solution

$\frac{\partial}{\partial t}u(x,t) = (2t+1)\cos(x)$, $\frac{\partial}{\partial x}u(x,t) = -(t^2+t)\sin(x)$ and $\frac{\partial^2}{\partial x^2}u(x,t) = -(t^2+t)\cos(x)$
Substituting these in the above heat equation, we get:
$f(x,t) = (t^2+3t+1)\cos(x)$

But one should not forget to observe that $u(x,t) = (t^2+t)\cos(x)$ also satisfies the boundary conditions and hence it is a solution.

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Exercise 3: *Classification of differential equations* **(9 points)**

Classify (order, linear/nonlinear, stationary/instationary, homogeneous, inhomogeneous) the following differential equations:

(a)

$$\frac{\partial^3 u}{\partial x^3} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} = 0$$

(4 points)

Solution

4th Order. Since it does not depend on time, it is stationary. Consider $u = 0$, and replace it in the Differential Equation, then:

$\frac{\partial^3 u}{\partial x^3} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} = 0$. Therefore, the equation is homogeneous.

Linearity condition: $L(\alpha u + \beta v) = \alpha L(u) + \beta L(v)$. In this case, $L(u) = \frac{\partial^3 u}{\partial x^3} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4}$.

Then, $L(\alpha u + \beta v) = \frac{\partial^3(\alpha u + \beta v)}{\partial x^3} + 2\frac{\partial^4(\alpha u + \beta v)}{\partial x^2 \partial y^2} + \frac{\partial^4(\alpha u + \beta v)}{\partial y^4}$

$= \alpha\frac{\partial^3 u}{\partial x^3} + \beta\frac{\partial^3 v}{\partial x^3} + 2\alpha\frac{\partial^4 u}{\partial x^2 \partial y^2} + 2\beta\frac{\partial^4 v}{\partial x^2 \partial y^2} + \alpha\frac{\partial^4 u}{\partial y^4} + \beta\frac{\partial^4 v}{\partial y^4}$

$= \alpha(\frac{\partial^3 u}{\partial x^3} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4}) + \beta(\frac{\partial^3 v}{\partial x^3} + 2\frac{\partial^4 v}{\partial x^2 \partial y^2} + \frac{\partial^4 v}{\partial y^4}) = \alpha L(u) + \beta L(v)$.

Consequently, the PDE is linear.

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(b)

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + \sin(u) = x\sin(t)$$

(5 points)

Solution

2nd Order, instationary PDE. Since u = 0 leads to a non-zero RHS value $x\sin(t)$, it is a non-homogeneous PDE.

Linearity condition: $L(\alpha u + \beta v) = \alpha L(u) + \beta L(v)$.
In this case, $L(u) = \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + \sin(u)$

Then, $L(\alpha u + \beta v) = \frac{\partial^2 (\alpha u + \beta v)}{\partial t^2} - \frac{\partial^2 (\alpha u + \beta v)}{\partial x^2} + \sin(\alpha u + \beta v)$

$= \alpha(\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2}) + \beta(\frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 v}{\partial x^2}) + \sin(\alpha u)\cos(\beta v) + \cos(\alpha u)\sin(\beta v) \neq \alpha L(u) + \beta L(v)$.

The term leading to non-linearity is $\sin(u)$

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Exercise 4: *Analytic solution to a PDE* **(20 points)**

Consider the PDE

$$u_t - c^2 u_{xx} = 0 \qquad \text{for } x \in (0, \pi) \text{ and } t \in (0, \infty)$$

with initial and Neumann boundary conditions

$$u(x,0) = \cos(2x) \qquad \frac{\partial u}{\partial x}(0,t) = 0, \qquad \frac{\partial u}{\partial x}(\pi,t) = 0.$$

(a) Do a separation–Ansatz and thus derive two separate ODEs

$$\frac{\dot{f}}{f} = c^2 \frac{g''}{g} = A, \qquad \text{with } A \in \mathbb{R}$$

(4 points)

Solution

The ansatz is:

$$u = f(t)g(x)$$

$u_t = \dot{f}(t)g(x)$ and $u_{xx} = f(t)g''(x)$

Substituting these in the given PDE, we get the two ODEs:

$$\text{ODE1: } \frac{\dot{f}}{f} = A$$

$$\text{ODE2: } c^2 \frac{g''}{g} = A$$

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(b) Solve both ODEs subject to the boundary conditions to get an infinite number of particular solutions of the PDE. You may assume that the separation–constant $A < 0$. Write down the general solution of the PDE without regard to the initial conditions as a sum (superposition) of all particular solutions.

Solution

Solve

$$c^2 g'' - Ag = 0$$

if we look for the function $g(x)$ in the form:

$$g(x) = e^{\lambda x}$$

$$c^2\lambda^2 e^{\lambda x} - Ae^{\lambda x} = 0$$

Dividing both sides by $e^{\lambda x}$, we get to the characteristic equation:

$$c^2\lambda^2 - A = 0$$

$$\lambda^2 = \frac{A}{c^2} \qquad \lambda = \pm\sqrt{\frac{A}{c^2}}$$

where λ is a complex number when $A < 0$ assumed:

$$\lambda = \pm i\sqrt{\frac{-A}{c^2}}$$

And thus the solution has the form:

$$g(x) = C_1 e^{i\sqrt{\frac{-A}{c^2}}x} + C_2 e^{-i\sqrt{\frac{-A}{c^2}}x}$$

Or in different form:

$$g(x) = B_1 \cos\left(\sqrt{\frac{-A}{c^2}}x\right) + B_2 \sin\left(\sqrt{\frac{-A}{c^2}}x\right)$$

To apply the Neumann boundary conditions, we need the derivative of the solution:

$$\frac{\partial u}{\partial x} = \frac{\partial(f(t)g(x))}{\partial x} = f(t)\frac{\partial g(x)}{\partial x}$$

$$\frac{\partial g(x)}{\partial x} = -B_1\sqrt{\frac{-A}{c^2}} \sin\left(\sqrt{\frac{-A}{c^2}}x\right) + B_2\sqrt{\frac{-A}{c^2}} \cos\left(\sqrt{\frac{-A}{c^2}}x\right)$$

$$\frac{\partial u}{\partial x}(0,t) = 0 \rightarrow u(x) \text{ only gives non-trivial solution if } g'(0) = 0$$

$$\frac{\partial u}{\partial x}(\pi,t) = 0 \rightarrow u(x) \text{ only gives non-trivial solution if } g'(\pi) = 0$$

$$g'(0) = -B_1\sqrt{\frac{-A}{c^2}} \underbrace{\sin\left(\sqrt{\frac{-A}{c^2}}0\right)}_{=0} + B_2\sqrt{\frac{-A}{c^2}} \underbrace{\cos\left(\sqrt{\frac{-A}{c^2}}0\right)}_{=1} \rightarrow B_2 = 0$$

Thus $g(x)$ has only cosine terms.

$$g'(\pi) = -B_1\sqrt{\frac{-A}{c^2}} \sin\left(\sqrt{\frac{-A}{c^2}}\pi\right) = 0 \rightarrow \sin\left(\sqrt{\frac{-A}{c^2}}\pi\right) = 0$$

$$\left(\sqrt{\frac{-A}{c^2}}\pi\right) = k\pi \quad k = 1, 2..n$$

$$\rightarrow \quad -A = (kc)^2 \quad k = 1, 2..n$$

And accordingly a sequence of solution for $g(x)$ is:

$$g_k(x) = B_{1k}\cos(kx)$$

Solving the first ODE (ODE1):

$$\dot{f} = Af \rightarrow \quad f(t) = De^{At} \rightarrow \quad f_k(t) = D_k e^{-(kc)^2 t}$$

Consequently, one sequence of the solution from the ansatz

$$u_k(x,t) = f_k(t)g_k(x) = C_k e^{-(kc)^2 t} \cos(kx)$$

And the solution:

$$u(x,t) = \sum_{k=1}^{\infty} u_k = \sum_{k=1}^{\infty} C_k e^{-(kc)^2 t} \cos(kx)$$

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(12 points)

(c) Incorporate the initial conditions to find the exact solution of the PDE.

Solution

Applying the initial condition:

$$u(x,0) = \cos(2x)$$

$$u(x,0) = \sum_{k=1}^{\infty} C_k \underbrace{e^{-(kc)^2 0}}_{=1} \cos(kx) = \sum_{k=1}^{\infty} C_k \sin(kx) = \cos(2x) \rightarrow \quad C_2 = 1, \quad C_i = 0 \quad \text{for } i \neq 2$$

$$u(x,t) = e^{-4c^2 t} \cos(2x)$$

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(4 points)