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Due date: 4. Nov. 2016

Introduction to PDEs and Numerical Methods (PDEs 1): Assignment 1 (50 points)

Exercise 1: *Differential operators* **(15 points)**

(a) Let $f_1(x, y) = ze^x \sin(y)$. Determine $\frac{\partial f_1}{\partial x}$, $\frac{\partial f_1}{\partial y}$, $\frac{\partial f_1}{\partial z}$ and ∇f_1. (4 points)

(b) Let $\mathbf{f_2}(x, y) = (xy^2, xy, \cos(z))^T$. Determine $\nabla \cdot \mathbf{f_2}$ and $\nabla \times \mathbf{f_2}$. (4 points)

(c) Let $f_3(x, y, z) = x^2 + y^4 z$. Determine Δf_3. (3 points)

(d) Show that $\nabla \cdot \nabla f = \Delta f$ and $\nabla \times \nabla f = 0$ for any two-times differentiable function $f : \Omega \to \mathbb{R}^3$. (4 points)

Exercise 2: *Heat equation* **(6 points)**

Consider the heat equation on a bar of unit length, with parameter β^2:

$$\frac{\partial}{\partial t} u(x, t) - \beta^2 \frac{\partial^2}{\partial x^2} u(x, t) = f(x, t)$$

(a) Assume boundary conditions $u(0, t) = 0$, $u(\pi, t) = 0$ and the source term $f(x, t) = 0$. Prove that $u(x, t) = e^{-2t} \sin(x)$ can be a solution of the heat equation and specify the value of β^2 that ensures this proof. (3 points)

(b) Now assume $\beta = 1$, boundary conditions $\frac{\partial u}{\partial x}(0, t) = 0$ and $\frac{\partial u}{\partial x}(\pi, t) = 0$ and a solution $u(x, t) = (t^2 + t) \cos(x)$. What must $f(x, t)$ look like if the heat equation should be satisfied. (3 points)

Exercise 3: *Classification of differential equations* **(9 points)**

Classify (order, linear/nonlinear, stationary/instationary, homogeneous, inhomogeneous) the following differential equations:

(a)

$$\frac{\partial^3 u}{\partial x^3} + 2 \frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} = 0$$

(4 points)

(b)

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + \sin(u) = x \sin(t)$$

(5 points)

Exercise 4: *Analytic solution to a PDE* **(20 points)**

Consider the PDE

$$u_t - c^2 u_{xx} = 0 \qquad \text{for } x \in (0,\pi) \text{ and } t \in (0,\infty)$$

with initial and Neumann boundary conditions

$$u(x,0) = \cos(2x) \qquad \frac{\partial u}{\partial x}(0,t) = 0, \qquad \frac{\partial u}{\partial x}(\pi,t) = 0.$$

(a) Do a separation–Ansatz and thus derive two separate ODEs

$$\frac{\dot{f}}{f} = c^2 \frac{g''}{g} = A, \qquad \text{with } A \in \mathbb{R}$$

(4 points)

(b) Solve both ODEs subject to the boundary conditions to get an infinite number of particular solutions of the PDE. You may assume that the separation–constant $A < 0$. Write down the general solution of the PDE without regard to the initial conditions as a sum (superposition) of all particular solutions.

(12 points)

(c) Incorporate the initial conditions to find the exact solution of the PDE.

(4 points)