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Assignment 4
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Introduction to Scientific Computing

Exercise 1: *Fixed points and stability for a nonlinear system of equations* **15 points**

(a) The Lorenz system (which is very famous in chaos theory) is given by (5 points)

$$\dot{x} = a(y-x),$$

$$\dot{y} = bx - y - xz,$$

$$\dot{z} = xy - cz,$$

where a, b and c are free parameters. Find the fixed points of the system. There should be three.

(b) The Hénon system (which is quite famous in non-linear dynamics/chaos theory) is given by: (5 points)

$$x_{n+1} = 1 - ax_n^2 + y_n$$

$$y_{n+1} = bx_n$$

Find the fixed points $(x_*, y_*)^{\mathrm{T}}$ of this system. Give a formula for x_* and y_* depending on a and b.

(c) Given the system

$$\dot{x} = \cos y - 0.75(x - \sin y), \quad \dot{y} = 1,$$

find the stable solutions. (5 points)

Exercise 2: *Jordan decomposition* **6 points**

Transform the matrix

$$A = \begin{pmatrix} 2 & -3 \\ 3 & -4 \end{pmatrix}$$

into the Jordan form, and compute the generalised eigenvector.

Exercise 3: *Equivalence of characteristic polynomials* **15 points**

Prove that the characteristic polynomial of a linear difference equation of order k, without loss of generality $a_0 = 1$, is the same as of matrix A from the corresponding difference equation system of dimension k and order 1.

Hint:

a) Given the linear differential equation

$$a_n x^{(n)} + a_{n-1} x^{(n-1)} + \ldots + a_0 x = 0,$$

one assumes that solutions to this differential equation will be in the form $x(t) = e^{\lambda t}$, then plugging this into the differential equation one gets

$$e^{\lambda t}(a_n\lambda^n + a_{n-1}\lambda^{n-1} + ... + a_1\lambda + a_0) = 0.$$

The polynomial $p(\lambda) = a_n\lambda^n + a_{n-1}\lambda^{n-1} + ... + a_1\lambda + a_0$ is called the characteristic polynomial of the above defined differential equation.

b) For the k-th order difference equation

$$x_{n+1} = -\sum_{i=1}^{k} a_i x_{n+1-i}$$

the corresponding first order system $\vec{x}_{n+1} = A\vec{x}_n$ is given by

$$\vec{x}_{n+1} = \begin{pmatrix} -a_1 & -a_2 & ... & & & -a_k & \\ 1 & 0 & ... & & & & \\ 0 & 1 & 0 & & & & \\ & & & & & & \\ & & & \ddots & & & \\ & & & ... & 1 & 0 & 0 \\ & & & ... & 0 & 1 & 0 \end{pmatrix} \vec{x}_n.$$

Calculate $\det(A - \lambda\,\mathrm{Id})$ for some n in a sensible way to see why the claim is valid, then do induction.