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# *Notes: Approximate Block Newton's method*

Suppose we have two coupled $m$-dimensional systems:

$$\begin{aligned} \mathcal{F}(\mathbf{x},\mathbf{y}) &= 0, \qquad \text{or re-written as:} \qquad & \mathbf{x} &= \boldsymbol{f}(\mathbf{x},\mathbf{y}) \\ \mathcal{G}(\mathbf{x},\mathbf{y}) &= 0, & \mathbf{y} &= \boldsymbol{g}(\mathbf{x},\mathbf{y}) \end{aligned} \tag{1}$$

with $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^m$, the re-writing can be made by, e.g. let $\boldsymbol{f} = \mathbf{x} - \mathcal{F}(\mathbf{x},\mathbf{y})$. And we have also the corresponding iterative solver of them:

$$\begin{aligned} \mathbf{x}^{(k+1)} &= \boldsymbol{f}(\mathbf{x}^{(k)},\mathbf{y}) \\ \mathbf{y}^{(k+1)} &= \boldsymbol{g}(\mathbf{y}^{(k)},\mathbf{x}) \end{aligned}$$

A monolithic Newtone's method solves the system

$$\begin{pmatrix} \mathbf{I}-\mathbf{f}_x & -\mathbf{f}_y \\ -\mathbf{g}_x & \mathbf{I}-\mathbf{g}_y \end{pmatrix} \begin{pmatrix} \Delta\mathbf{x} \\ \Delta\mathbf{y} \end{pmatrix} = - \begin{pmatrix} \mathbf{x}-\boldsymbol{f}(\mathbf{x},\boldsymbol{y}) \\ \mathbf{y}-\boldsymbol{g}(\mathbf{x},\boldsymbol{y}) \end{pmatrix}$$

where the $\mathbf{f}_x$ is the Jacobi of $\boldsymbol{f}$ with respect to the vector $\mathbf{x}$ (and similar for $\mathbf{f}_y$, $\mathbf{g}_x$ and $\mathbf{g}_y$).
To avoid solving the global system we use a block Gauss elimination in stead, we follows the procedure introduced in [1] which consists of the following steps:

1 Solve $(\mathbf{I}-\mathbf{f}_x)\mathbf{q} = \boldsymbol{f}(\mathbf{x},\mathbf{y}) - \mathbf{x}$ for $\mathbf{q}$.

2 solve $(\mathbf{I}-\mathbf{f}_x)\mathbf{C} = \mathbf{f}_y$ for $\mathbf{C}$,
and solve $(\mathbf{I}-\mathbf{g}_y-\mathbf{g}_x\mathbf{C})\Delta\mathbf{y} = \boldsymbol{g}(\mathbf{x},\mathbf{y}) + \mathbf{g}_x\mathbf{q} - \mathbf{y}$ for $\Delta\mathbf{y}$.

3 $\Delta\mathbf{x} = \mathbf{q} + \mathbf{C}\Delta\mathbf{y}$

This procedure(block Newton) solves three $m$-by-$m$ systems in stead of the $2m$-by-$2m$ global system. The derivation of the procedure is as introduced in the tutorial.

The cross-derivative $\mathbf{f}_y$ and $\mathbf{g}_x$ is usually not available in the context of partitioned methods. But since what we really need are only the product $\mathbf{f}_y\mathbf{u}$ or $\mathbf{g}_x\mathbf{u}$ ($\boldsymbol{u}$ some vector), the product can be approximated by finite differences. This leads to an approximate block Newton method. The finite differences are made in the following way:

- In the 3rd step of the above procedure, we need to evaluate $\mathbf{f}_y\Delta\mathbf{y}$, this is to be approximated by:

$$\mathbf{f}_y\Delta\mathbf{y} \approx \frac{1}{h}(\boldsymbol{f}(\mathbf{x},\mathbf{y}+h\Delta\mathbf{y}) - \boldsymbol{f}(\mathbf{x},\mathbf{y})),$$

  $h$ is a small value, so that $\mathbf{C}\Delta\mathbf{y}$ can be computed without knowing $\mathbf{f}_y$.

- In the 2nd step, $\boldsymbol{g}(\mathbf{x},\mathbf{y}) + \mathbf{g}_x\mathbf{q}$ is approximated by

$$\boldsymbol{g}(\mathbf{x},\mathbf{y}) + \mathbf{g}_x\mathbf{q} \approx \boldsymbol{g}(\mathbf{x}+\mathbf{q},\mathbf{y})$$

- In the 2nd step, $(\mathbf{I} - \mathbf{g}_y - \mathbf{g}_x\mathbf{C})\Delta\mathbf{y}$ is approximated by

$$
\begin{aligned}
(\mathbf{I} - \mathbf{g}_y - \mathbf{g}_x\mathbf{C})\Delta\mathbf{y} &= \frac{1}{h}\left[(\mathbf{I} - \mathbf{g}_y)h\Delta\mathbf{y} - \mathbf{g}_x\mathbf{C}h\Delta\mathbf{y}\right] \\
&\approx \frac{1}{h}\left[h\Delta\mathbf{y} + \boldsymbol{g}(\mathbf{x} - \mathbf{C}h\Delta\mathbf{y}, \mathbf{y} - h\Delta\mathbf{y}) - \boldsymbol{g}(\mathbf{x}, \mathbf{y})\right]
\end{aligned}
$$

  The $\mathbf{C}h\Delta\mathbf{y}$ is to be computed as in the above approximation in the 3rd step.

  In this step, the solution of $\Delta\mathbf{y}$ can be made by a biconjugate gradient solver which solves $\mathbf{Ax} = \mathbf{b}$ without knowing $\mathbf{A}$ but instead taking a function which returns $\mathbf{Au}$ at every input $\mathbf{u}$. This function can be defined by using the above finite difference approximation.

With these approximations the block Newton method can be carried on without the cross-derivatives.

## References


[1] H. G. Matthies and J. Steindorf. Strong coupling methods. In W. Wendland and M. Efendiev, editors, *Analysis and Simulation of Multifield Problems*, pages 13–36, 2003.