Hence,

$$\frac{h^2}{2} f_t = b_2 c_2 f_t \Rightarrow \boxed{b_2 c_2 = 1/2}$$

and

$$\frac{h^2}{2} f_x f = h^2 b_2 a_{21} f_x f \Rightarrow b_2 a_{21} = 1/2$$

Let us take $b_1 = \frac{1}{4} \Rightarrow b_2 = 3/4$

$$b_2 c_2 = 1/2, \quad \frac{3}{4} c_2 = \frac{1}{2}, \quad \Rightarrow c_2 = 2/3$$

and

$$b_2 a_{21} = 1/2 \Rightarrow a_{21} = 2/3$$

Ex2. Please, see attached code for the general Runge-Kutta method.

c) To estimate the order, one may use the method of halving step size.

$$\varepsilon_{loc}^{h} = ch^p \qquad \varepsilon_{loc}^{h/4} = c\left(\frac{h}{4}\right)^p$$

$$\varepsilon_{loc}^{h/2} = c\left(\frac{h}{2}\right)^p \qquad \varepsilon_{loc}^{h/8} = c\left(\frac{h}{8}\right)^p$$

$$\frac{\varepsilon_{loc}^{h} - \varepsilon_{loc}^{h/2}}{\varepsilon_{loc}^{h/2} - \varepsilon_{loc}^{h/4}} = \frac{ch^p - c\left(\frac{h}{2}\right)^p}{c\left(\frac{h}{2}\right)^p - c\left(\frac{h}{4}\right)^p} =$$

$$= \frac{\cancel{ch^p}\left(1 - \frac{1}{2^p}\right)}{\cancel{ch^p}\left(\frac{1}{2^p} - \frac{1}{2^{2p}}\right)} = \frac{\cancel{1 - \frac{1}{2^p}}}{\frac{1}{2^p}\left(\cancel{1 - \frac{1}{2^p}}\right)} = 2^p$$

$$\varepsilon_{loc}^{h} - \varepsilon_{loc}^{h/2} = \cancel{x_n^t} - x_n^h - (\cancel{x_n^t} - x_n^{h/2})$$
$$= x_n^{h/2} - x_n^h$$

Hence, $2^p = \dfrac{x_n^{h/2} - x_n^{h}}{x_n^{h/4} - x_n^{h/2}} \Rightarrow p = \ldots$

Ex1.

Solution

$$
\begin{array}{c|cc}
0 & 0 & 0 \\
\frac{2}{3} & \frac{2}{3} & 0 \\
\hline
 & \frac{1}{4} & \frac{3}{4}
\end{array}
$$

Runge-Kutta method (explicit):

$$K_i = f\left(t_n + c_i h,\ x_n + h \sum_{j=1}^{i} a_{ij} K_j\right)$$

$$x_{n+1} = x_n + h \sum_{j=1}^{s} b_j K_j$$

Local error:

$$\varepsilon_{loc} = x_{n+1}^t - x_{n+1} =$$

$$= \left(\cancel{x_n^t} + h f(t_n, x_n^t) + \frac{h^2}{2} \dot{f}(t_n, x_n^t) + \frac{h^3}{6} \ddot{f}(t_n, x_n^t)\right) - \left(\cancel{x_n^t} + h \sum b_j f\left(t_n + c_i h,\ x_n^t + h \sum_{j=1}^{i} a_{ij} K_j\right)\right) =$$

$$= h f(t_n, x_n^t) + \frac{h^2}{2} \dot{f}(t_n, x_n^t) + \frac{h^3}{6} \ddot{f}(t_n, x_n^t)$$

$$- h b_1 f\big(t_n + \underbrace{\cancel{c_1}}_{\text{Explicit RK}} h,\ x_n^t + h a_{11} \underbrace{K_1}_{\Downarrow\ f(t_n, x_n^t)} + h \underbrace{\cancel{a_{12} K_2}}_{\text{Explicit RK}}\big)$$

$$- h b_2 f\big(t_n + c_2 h,\ x_n^t + h a_{21} f(t_n, x_n^t) + h \underbrace{\cancel{a_{22} K_2}}_{\text{Explicit RK}}\big)\big)$$

| ASS1

$$\varepsilon_{loc} = hf(t_n, x_n^t) + \frac{h^2}{2}\mathring{f}(t_n, x_n^t) + \frac{h^3}{6}\ddot{f}(t_n, x_n^t)$$
$$- hb_1 f(t_n, x_n^t + ha_{11} f(t_n, x_n^t)) - hb_2 f(t_n + c_2 h,$$
$$x_n^t + ha_{21} f(t_n, x_n^t))$$

Here,

$$\mathring{f}(t, x) = \frac{d}{dt} f(t, x) = f_t + \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} = f_t + f_x \cdot \dot{x} =$$
$$= f_t + f_x \cdot f$$

$$f(t_n, x_n^t + ha_{11} f(t_n, x_n^t)) =$$
$$= f(t_n, x_n^t) + f_x(t_n, x_n^t) \cdot (x_n^t + h\cancel{a_{11}}^{0} f(t_n, x_n^t)$$
$$- x_n^t) = f(t_n, x_n^t)$$

$$f(t_n + c_2 h, x_n^t + ha_{21} f(t_n, x_n^t)) =$$
$$= f(t_n, x_n^t) + f_t(t_n, x_n^t) c_2 h + f_x(t_n, x_n^t) h a_{21}$$
$$f(t_n, x_n^t)$$

$$\varepsilon_{loc} = hf \quad + \frac{h^2}{2}(f_t + f_x f) + \frac{h^3}{6}\ddot{f}$$
$$- hb_1 f \quad - hb_2 f \quad - hb_2 f_t c_2 h -$$
$$- hb_2 f_x h a_{21} f$$

$$|\varepsilon_{loc}| \leq Ch^3$$

Thus, each term with power less than 3 must be zero.

$$hf - hb_1 f - hb_2 f = 0 \Rightarrow \boxed{b_1 + b_2 = 1}$$

$$\frac{h^2}{2}(f_t + f_x f) - h^2 b_2 c_2 f_t - h^2 b_2 a_{21} f_x f = 0$$