Exact Recovery of Partially Sparse Vectors

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Problem

We consider the general problem

$$\min_{x\in\mathbb{R}^n} \quad f(x_1,x_2) \quad \text{s.t.} \quad A_1x_1 + A_2x_2 = b$$

x_1 is assumed to be sparse, x_2 is expected to be dense

Mixed Norms

As objective function, we consider

- the weighted ℓ_1-ℓ_2-norm

$$\|x\|_{M,\alpha} := \|x_1\|_1 + \alpha\|x_2\|_2$$

- the Luxemburg norm

$$\|x\|_{L,\beta} := \inf\left\{\lambda > 0 \,:\, \|\tfrac{x_1}{\lambda}\|_1 + \beta\|\tfrac{x_2}{\lambda}\|_2^2 \le 1\right\} = \tfrac{1}{2}\|x_1\|_1 + \sqrt{\tfrac{1}{4}\|x_1\|_1^2 + \beta\|x_2\|_2^2}$$

Relationship

- $\|x\|_{M,\alpha} = \|x\|_{L,\beta}$ if $x_1 = 0$ and $\beta = \alpha^2$ (or if $x_2 = 0$)
- $\beta = \alpha^2$ minimizes ratio of largest and smallest values of $\|\cdot\|_{L,\beta}$ on the unit sphere w.r.t. $\|\cdot\|_{M,\alpha}$

Recovery Conditions

Theorem (weighted ℓ_1-ℓ_2-norm case):
A point x^* with $x_2^* \neq 0$ is a solution of

$$\min_{x\in\mathbb{R}^n} \quad \|x\|_{M,\alpha} \quad \text{s.t.} \quad Ax = Ax^*$$

if and only if there exists $w^* \in \mathbb{R}^m$ such that

$$(A_1)^\top w^* \in \partial\|x_1^*\|_1 \qquad (1)$$

$$\text{and} \quad (A_2)^\top w^* = \frac{\alpha}{\|x_2^*\|_2}\cdot x_2^*. \qquad (2)$$

Theorem (Luxemburg norm case):
A point $x^* \neq 0$ is a solution of

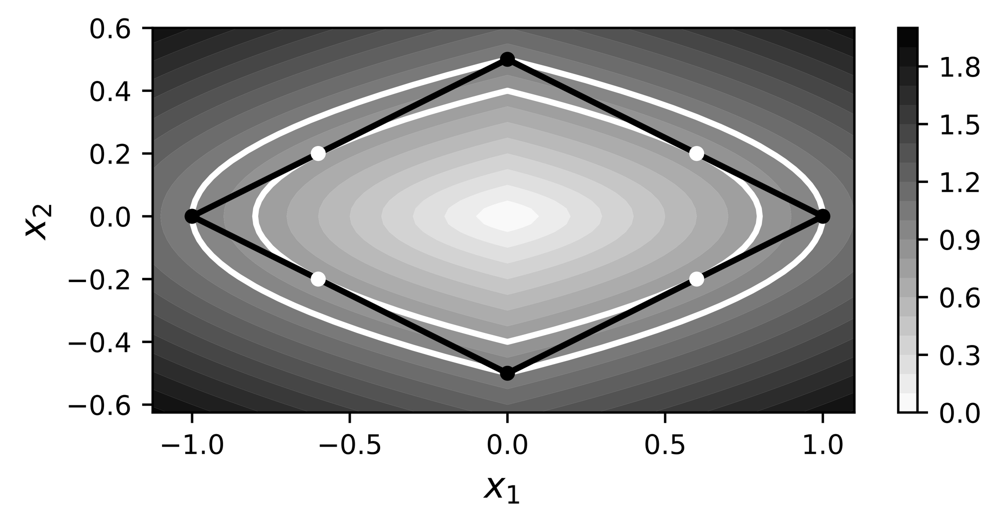
$$\min_{x\in\mathbb{R}^n} \|x\|_{L,\beta} \quad \text{s.t.} \quad Ax = Ax^*$$

if and only if there exists $w^* \in \mathbb{R}^m$ such that

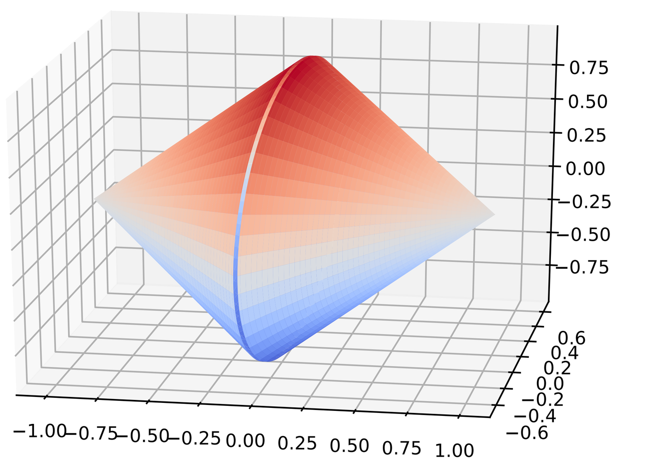
$$(A_1)^\top w^* \in \left(\frac{1}{2} + \frac{\|x_1^*\|_1}{4\sqrt{\frac{1}{4}\|x_1^*\|_1^2 + \beta\|x_2^*\|_2^2}}\right)\partial\|x_1^*\|_1 \qquad (3)$$

$$\text{and} \quad (A_2)^\top w^* = \left(\frac{\beta}{\sqrt{\frac{1}{4}\|x_1^*\|_1^2 + \beta\|x_2^*\|_2^2}}\right)\cdot x_2^*. \qquad (4)$$

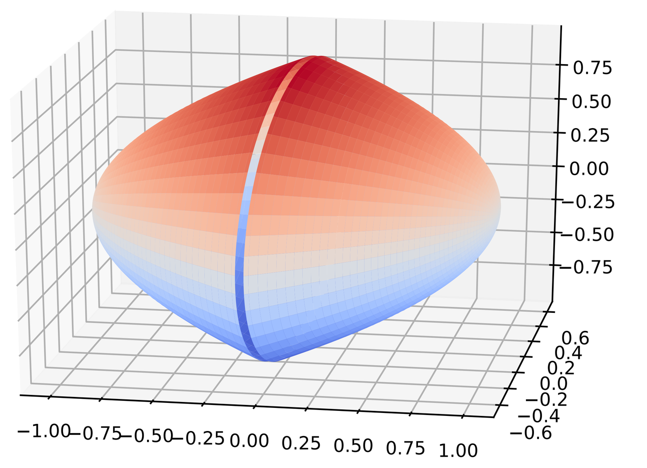
Comparison of Unit Balls



unit balls of $\|\cdot\|_{M,2}$ (black), $\|\cdot\|_{L,4}$ (outer white), and $\|\cdot\|_{L,4}$-ball of radius $4/5$ (inner white)

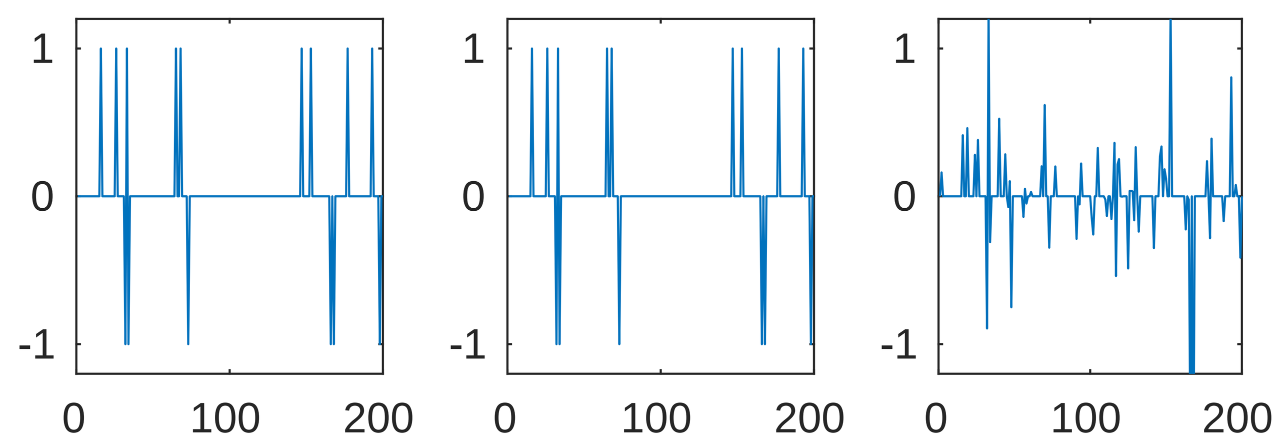


3D-unit ball of $\|\cdot\|_{M,1}$ ($x_2 \in \mathbb{R}^2$)

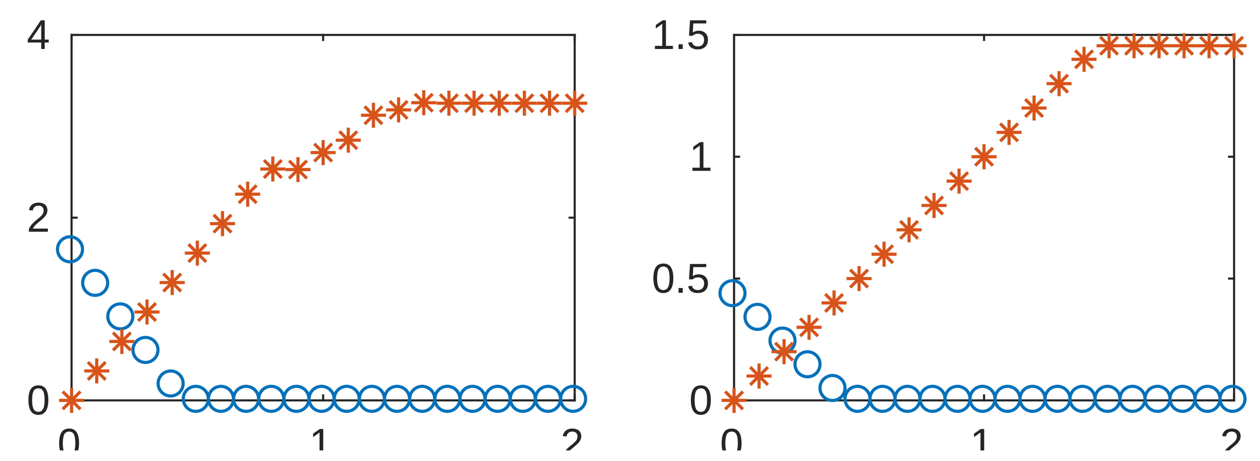


3D-unit ball of $\|\cdot\|_{L,1}$ ($x_2 \in \mathbb{R}^2$)

Computational Experiments



Reconstruction from $b := A_1x_1 + A_2t$ with Gaussian $A_1 \in \mathbb{R}^{60\times 200}$, $A_2 \in \mathbb{R}^{60}$ vector of all ones, some sparse x_1^* (left plot), and $t = 2$. Middle: recovered x_1 by the Luxemburg norm $\|\cdot\|_{L,\alpha^2}$, right: recovered x_1 by the mixed norm $\|\cdot\|_{M,\alpha}$, each with $\alpha = 4$.



Recovery error in the same setting as in leftmost figure. Left: error $\|x_1^* - x_1\|_2$ recovered with $\|\cdot\|_{L,16}$ (blue circles) and $\|\cdot\|_{M,4}$ (red stars) as a function of t^*. Right: error $|t^* - t|$ as a function of t^*. We conclude that the Luxemburg norms recovers x_1^* and t^* exactly for t^* large enough, the mixed norm for $t^* = 0$ only.

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