One-shot Multiple Access Channel Simulation

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Abstract

We consider the problem of shared randomness-assisted multiple access channel (MAC) simulation for product inputs and characterize the one-shot communication cost region via almost-matching inner and outer bounds in terms of the smooth max-information of the channel, featuring auxiliary random variables of bounded size. The achievability relies on a rejection-sampling algorithm to simulate an auxiliary channel between each sender and the decoder, and producing the final output based on the output of these intermediate channels. The converse follows via information-spectrum based arguments. To bound the cardinality of the auxiliary random variables, we employ the perturbation method from [Anantharam *et al.*, IEEE Trans. Inf. Theory (2019)] in the one-shot setting. For the asymptotic setting and vanishing errors, our result expands to a tight single-letter rate characterization and consequently extends a special case of the simulation results of [Kurri *et al.*, IEEE Trans. Inf. Theory (2022)] for fixed, independent and identically distributed (iid) product inputs to universal simulation for any product inputs.

We broaden our discussion into the quantum realm by studying feedback simulation of quantum-to-classical (QC) MACs with product measurements [Atif *et al.*, IEEE Trans. Inf. Theory (2022)]. For fixed product inputs and with shared randomness assistance, we give a quasi tight one-shot communication cost region with corresponding single-letter asymptotic iid expansion.

I. Motivation

The channel simulation problem deals with the task of quantifying the minimum amount of communication required to establish correlation remotely, as dictated by the input-output joint distribution of the channel to be simulated. The most basic point-to-point channel simulation setup consists of an encoder-decoder pair that with access to shared randomness and communication over a noiseless rate-limited link achieves the channel simulation task. More specifically, the encoder observes a random variable, say X with distribution q_X, and based on the shared randomness, sends a message to the decoder. Based on this message and shared randomness, the decoder outputs a random variable Y. The aim of the protocol is to ensure that the trace distance between the joint distribution (X, Y) and the joint distribution induced by passing the source X through a discrete memoryless channel $q(Y|X)$ is as small as possible. The channel simulation task is closely related to the task of creating a desired joint distribution between two distributed parties, also known as strong coordination [1].

Here, we consider the problem of simulating a two-sender classical and quantum to classical multiple-access channel (MAC). We assume that the respective encoders and decoder have access to unlimited shared randomness. This framework was first investigated by Bennett *et al.* [2] to establish a so-called 'reverse Shannon theorem' to simulate a noisy channel from a noiseless channel in the asymptotic independent and identically distributed (iid) regime. They showed that the least communication cost for this purpose is equal to the mutual information, $I(X;Y)$, between the input and output of the channel. The minimum one-shot rate for simulating a point-to-point classical channel was ascertained in [3]. Extensions to broadcast channels were obtained in [3], and recently extended to the quantum setting [4].

In both the point-to-point and broadcast channel simulation tasks, one may gain intuition from the scheme achieving the minimal communication rate as follows. Consider the case of point-to-point channel simulation: since both the encoder, say Alice, and the decoder, Bob, knows the channel to be simulated, Alice can determine the channel output at her end and then compress it 'optimally' and send it to Bob using the rate limited link. Bob then just outputs the target sequence after decompressing what he received from Alice. Similar intuition also works for the broadcast channel simulation problem. However, this approach breaks down for the MAC since there are two senders involved. More specifically, although each sender knows the MAC to be simulated, they cannot "locally" simulate the channel since the input of the other sender is unknown. Hence, novel schemes are required to circumvent this technical hurdle, which we address in this work. As such, While there is a comprehensive literature on simulating a point-to-point channel (both classical and quantum, in one-shot and asymptotic iid setting) and broadcast channel, only some more restricted results are known for MACs. In this regard, bounds on the asymptotic rate region for MAC simulation with fixed iid inputs were previously given in [5]. The inner bounds were derived by using the so-called OSRB technique of Yassaee *et al.* [6] and a matching outer bound for the case of fixed iid product inputs and shared randomness assistance was proven by using the continuity property of the mutual information.

Our main results are as follows:

- We obtain the one-shot cost region for simulating a MAC with two independent classical inputs (X_1, X_2) and a single classical output Y, where the MAC is represented by the conditional probability distribution $q_{Y|X_1,X_2}$. We characterize the cost region, first for fixed product inputs in Theorem 1, and then for universal simulation with arbitrary product inputs in Theorem 2. In order to simulate $q_{Y|X_1,X_2}$, for $j \in \{1, 2\}$, encoder $\mathcal{E}_j$ of the sender j sends a message $M_j \in [1 : 2^{R_j}]$ to the decoder $\mathcal{D}$ over their respective noiseless links based on their individual observations and shared randomness with the decoder. We assume unlimited shared randomness S_1 ($|\mathcal{S}_1| = \infty$) between $\mathcal{E}_1$ and $\mathcal{D}$ and S_2 ($|\mathcal{S}_2| = \infty$) between $\mathcal{E}_2$ and $\mathcal{D}$. Since there is neither a one-shot nor a universal analogue of the Yassaee *et al.* [6] OSRB techniques with the desired one-shot entropic quantity — which happens to be *smoothed max mutual information* in our case — this makes ours the first work towards simulating a MAC in the one-shot and universal regime.
- We specialize our result to the asymptotic iid setting and show that it recovers [5, Theorem 1] for fixed product iid inputs as Corollary 1.1 of Theorem 1, whereas we obtain a new single-letter formula for the universal case of arbitrary (not necessarily iid) product inputs in Corollary 2.1.
- We tightly characterize the one-shot cost and asymptotic rate region for simulating classical scrambling quantum-inputs and classical output MACs with feedback in Theorem 3 and Corollary 3.1, respectively. This is referred to as classical scrambling QC-MAC with feedback, where feedback denotes the property that the classical inputs to the scrambler should also be available at the sender(s) after the simulation protocol has been executed.

Technical contributions: A general recipe to obtain an inner bound on the rate region is applying the simple yet widely applicable technique of *rejection sampling*. One of the main technical hurdles, besides unavailability of both the inputs at the encoders, preventing the import of earlier results is that this task cannot be seen as naively carrying out two point-to-point channel simulation. The main reason is the fact that the output must be correlated with both the inputs. This is resolved by defining appropriate *auxiliary* random variables, which are quantized versions of the respective inputs such that they approximately simulate the channel. The distribution of these auxiliary random variables can equivalently be viewed as point-to-point channel and hence we use these to decompose MAC into two point-to-point channels. We also give the bounds on the cardinality of these auxiliary random variables in the one-shot setting for smoothed mutual information, which is rarely studied like [7] and the only known work to the best of our knowledge. But their technique of the so-called generalized support lemma does not suffice for the task of MAC simulation due to an extra requirement of preserving the property that the output should be generated in correlation with auxiliary random variables. Hence we apply for the first time, the perturbation technique developed by Anantharam *et al.* [8] for obtaining the cardinality bounds on auxiliary random variables for the smoothed max-mutual information.

II. Notation

The random variables are denoted by capital letters and their alphabets by scripted letters, for example, X is a random variable with alphabet $\mathcal{X}$, distributed according to p_X. p_X is also the probability vector with the set of non-zero entries denoting its support represented by $supp(p_X)$. Analogously, we denote any finite dimensional Hilbert space for quantum setting by $\mathcal{H}$. For brevity of notation, we use $\vec{X}$ to denote a finite length sequence of random variables $\{X_i\}_{i<\mathcal{I}}$, where $\mathcal{I}$ is any index set. The notation $[1 : n]$ is used as a shorthand to denote the discrete set $\{1, 2, \ldots, n\}$. Expectation of a random variable is denoted by $\mathbb{E}$. We use the abbreviation p.m.f. to mean the probability mass function of the underlying discrete valued random variable. The set of all probability vectors is denoted by $\mathcal{P}$ and sub-distribution vectors by $\mathcal{P}_{\leq}$ and analogously the set of all density operators in $\mathcal{H}$ is denoted by $\mathcal{D}$ or $\mathcal{D}(\mathcal{H})$ and sub-states, that is, positive semi-definite operators with trace less than or equal to 1 by $\mathcal{D}_{\leq}$. For $p, q \in \mathcal{P}$, the notation $p \ll q$ means that $supp(p) \subseteq supp(q)$. $h_2(\varepsilon)$ is the binary entropy of distribution $\{\varepsilon, 1-\varepsilon\}$, for $\varepsilon \in (0,1)$. It is defined as $h_2(\varepsilon) := -\varepsilon \log_2 \varepsilon - (1-\varepsilon)\log_2(1-\varepsilon)$. We use the notation $\|\cdot\|_1$ to denote the ℓ_1 norm of a vector, which is the sum of absolute value of its components and it denotes the Schatten 1-norm of the underlying operator. We use $\|x-y\|_{tvd} := \frac{\|x-y\|_1}{2}$ to denote the total variation distance between two vectors or operators. We define the joint distribution over a set of random variables by small case letters with the subscript denoting the random variables and the distribution restricted to a subset of random variables denotes their marginal. For example $p_{X_1,X_2,\ldots,X_n}$ denotes a joint distribution on the random variables $X_1, X_2, \ldots, X_n$ and p_{X_1,X_2} denotes the marginal on X_1, X_2 (by summing over the random variables $X_3, X_4, \ldots, X_n$). For brevity of notation we also define $p_{\vec{X}} := p_{X_1,X_2,\ldots,X_n}$. The notation $X \sim q$ indicates that the random variable X is distributed according to p or the p.m.f. of X is p and $X \overset{\varepsilon}{\sim} q$ means that the p.m.f. p of X is ε-close to the p.m.f. q in the total variation distance. We also use the notation $p \overset{\varepsilon}{\approx} q$ to denote that $\|p-q\|_{tvd} \leq \varepsilon$. We use the notation $p \ll q$, to mean that the p.m.f. p is absolutely continuous with respect to the p.m.f. q, with emphasis on the property that the $supp(p) \subseteq supp(q)$. The notation $\mathbb{1}A$ denotes the indicator random variable which takes a value 1 if event A occurs and is zero otherwise. Scripted letters denotes the encoders, decoders and channels. $cl\{\mathcal{S}\}$ denotes the closure of the set $\mathcal{S}$. For conditional distributions $p_{Y|X}$ and $q_{Y|X}$, the total variation distance is defined as $\left\|p_{Y|X} - q_{Y|X}\right\|_{tvd} := \max_x \left\|p_{Y|X=x} - q_{Y|X=x}\right\|_{tvd}$. We use tvd as our distance measure unless stated otherwise. All the alphabets and the dimensions of classical and/or quantum systems are finite. The notation $A \cong B$ is used to mean that the systems A and B are isomorphic to each other.

We now give the definitions of the entropic quantities used in this work.

Definition 1: The max divergence between any two probability distributions p and q on the support $\mathcal{X}$ is defined as

$$D_{\max}(p||q) \quad := \quad \log \max_{x \in \mathcal{X}} [p(x)/q(x)]$$

One then defines the notion of max-mutual information of a joint distribution $p_{X,Y}$ from $D_{\max}$ as follows.

Definition 2: For a given bipartite distribution $p_{X,Y}$ the max-mutual information is defined as [9]

$$I_{max}(X;Y)_p := \inf_{q_Y \in \mathcal{P}(\mathcal{Y})} D_{\max}(p_{X,Y}||p_X \times q_Y) = \inf_{q_Y \in \mathcal{P}(\mathcal{Y})} \max_x D_{\max}(p_{Y|X=x}||q_Y) \ .$$

and the ε-smoothed max-mutual information of $p_{X,Y} := p_X p_{Y|X}$ for $\varepsilon \geq 0$ is defined as [10]

$$I^{\varepsilon}_{\max}(X;Y)_p := \inf_{p'_{X,Y} \in \mathcal{B}^{\varepsilon}(p_{X,Y})} I_{\max}(X;Y)_{p'} = \inf_{p'_{X,Y} \in \mathcal{B}^{\varepsilon}(p_{X,Y})} \inf_{q_Y \in \mathcal{P}(\mathcal{Y})} \max_{x,y} \log \frac{p'_{Y|X}(y|x)}{q_Y(y)} \tag{1}$$

where $\mathcal{B}^{\varepsilon}(p_{X,Y}) := \{p'_{X,Y} \in \mathcal{P} : p'_X = p_X \text{ and } \mathop{\mathbb{E}}_{p_X} \|p_{Y|X} - p'_{Y|X}\|_{tvd} \leq \varepsilon\}$.

We now define the smoothed max-mutual information of channel using the notion of max-mutual information defined above.

Definition 3 (Channel smoothed max*-mutual information):* Let $p_{Y|X}$ denote a channel with input X and output Y. Let $X \sim p_X$ be a given input. Then for any given $\varepsilon > 0$: The ε-smoothed max-mutual information of the channel $p_{Y|X}$ is defined as [11]:

$$I^{\varepsilon}_{\max}(p_{Y|X}) := \inf_{p'_{Y|X} \in \mathcal{B}^{\varepsilon}(p_{Y|X})} \inf_{q_Y \in \mathcal{P}(\mathcal{Y})} \max_x D_{\max}(p'_{Y|X=x}||q_Y) = \inf_{p'_{Y|X} \in \mathcal{B}^{\varepsilon}(p_{Y|X})} \inf_{q_Y \in \mathcal{P}(\mathcal{Y})} \max_{x,y} \log \frac{p'_{Y|X}(y|x)}{q_Y(y)} \tag{2}$$

where $\mathcal{B}^{\varepsilon}(p_{Y|X}) := \{p'_{Y|X} \in \mathcal{P} : \max_x \|p_{Y|X=x} - p'_{Y|X=x}\|_{tvd} \leq \varepsilon\}$.

One defines the analogous smoothed max-mutual information for bipartite quantum states as follows [10]:

Definition 4 (State smoothed max*-mutual information):* The smoothed quantum max-mutual information is defined as

$$I^{\varepsilon}_{\max}(A;B)_{\rho} := \inf_{\rho'^{AB} \in \mathcal{B}^{\varepsilon}(\rho^{AB})} \inf_{\sigma^B \in \mathcal{D}(\mathcal{H}^B)} D_{\max}(\rho'^{AB} || \rho^A \otimes \sigma^B), \tag{3}$$

where

$$D_{\max}(\rho||\sigma) := \inf\{\lambda : \rho \leq 2^{\lambda}\sigma\} = \log\|\sigma^{-1/2}\rho\sigma^{-1/2}\|_{\infty}$$

and $\mathcal{B}^{\varepsilon}(\rho^{AB}) := \left\{\rho'^{AB} \in \mathcal{D}(\mathcal{H}) : \mathrm{Tr}_B(\rho') = \mathrm{Tr}_B(\rho), \|\rho - \rho'\|_{tvd} \leq \varepsilon\right\}$.

In the following, we use the term rate region or cost or cost region interchangeably to mean the amount of classical communication used (or charged for) in the simulation protocol and formally is the set of rate tuples (R_1, R_2) that ensures MAC simulation.

III. One-shot cost region for fixed product input MAC simulation

A. Task

We start by giving the formal definition of a 2-user MAC channel simulation code, described in Figure III-A.

Definition 5 (Classical MAC simulation with fixed input): An (R_1, R_2, ε) simulation protocol for a 2-independent user MAC $q_{Y|X_1X_2}$ with inputs $q_{X_1} \times q_{X_2}$ and access to unlimited shared randomness between Sender1$\overset{S_1}{\leftrightarrow}$ Receiver and Sender2$\overset{S_2}{\leftrightarrow}$ Receiver, consists of:

- A pair of encoders of form $\mathcal{E}_1 \times \mathcal{E}_2$, such that: $\mathcal{E}_j : \mathcal{X}_j \times \mathcal{S}_j \to \mathcal{M}_j := \left[1 : 2^{R_j}\right]$ for $j \in \{1, 2\}$;
- Two independent noiseless rate-limited links of rate R_j, $j \in \{1, 2\}$ and;
- A decoder $\mathcal{D} : \mathcal{M}_1 \times \mathcal{S}_1 \times \mathcal{M}_2 \times \mathcal{S}_2 \to \mathcal{Y}$;
- The overall joint distribution induced by the encoder-decoder pair is given by

$$p_{X_1,X_2,S_1,S_2,M_1,M_2,Y} = \left[\mathcal{D} \circ \left(\bigtimes_{j=1}^{2} \mathcal{E}_j\right)\right] \left\{\bigtimes_{j=1}^{2} \left(q_{X_j} \times p_{S_j}\right)\right\}$$

s.t.

$$\|p_{X_1,X_2,Y} - q_{X_1,X_2,Y}\|_{tvd} = \mathop{\mathbb{E}}_{q_{X_1} \times q_{X_2}} \left\|p_{Y|X_1,X_2} - q_{Y|X_1,X_2}\right\|_{tvd} \leq \varepsilon \ . \tag{4}$$

The rate region $\mathcal{R}(\varepsilon)$ for simulating MAC is defined as the closure of the set of all rate pairs (R_1, R_2) as given above satisfying (4).

In this section, we henceforth consider $p_{X_1,X_2,U_1,U_2,Y}$ to be a p.m.f. of the form:

$$p_{X_1,X_2,U_1,U_2,Y} = q_{X_1}q_{X_2}p_{U_1|X_1}p_{U_2|X_2}p_{Y|U_1,U_2}. \tag{5}$$

We now define the following regions that will turn out to be inner and outer bounds for characterizing the rate region $\mathcal{R}(\varepsilon)$ for the task of one-shot MAC simulation given in Definition 5.

Definition 6: Let $\varepsilon \in (0, 1)$, and $\varepsilon_1, \varepsilon_2, \delta \in (0, 1)$ be such that $\delta < \min\{\varepsilon_1, \varepsilon_2\}$ and $\varepsilon_1 + \varepsilon_2 \leq \varepsilon$. Let $\mathcal{R}_{inner}(\varepsilon_1, \varepsilon_2, \delta)$ be the set of non-negative real numbers (R_1, R_2) defined as:

$$\mathcal{R}_{inner}(\varepsilon_1, \varepsilon_2, \delta) = cl \left\{ \bigcup_{(p_{U_1|X_1}, p_{U_2|X_2}) \in \mathcal{A}^{inner}} \left\{ (R_1, R_2) : R_j \geq I^{\varepsilon_j - \delta}_{\max}(X_j; U_j)_{q_{X_j} p_{U_j|X_j}} + \log\log\frac{1}{\delta} \ ; j \in \{1, 2\} \right\} \right\}, \tag{6}$$

where

$$\mathcal{A}^{inner} := \left\{ \left(p_{U_1|X_1}, p_{U_2|X_2}\right) : \ \exists p_{Y|U_1,U_2} \text{ satisfying } \ p_{X_1,X_2,Y} = q_{X_1,X_2,Y} \right\}. \tag{7}$$

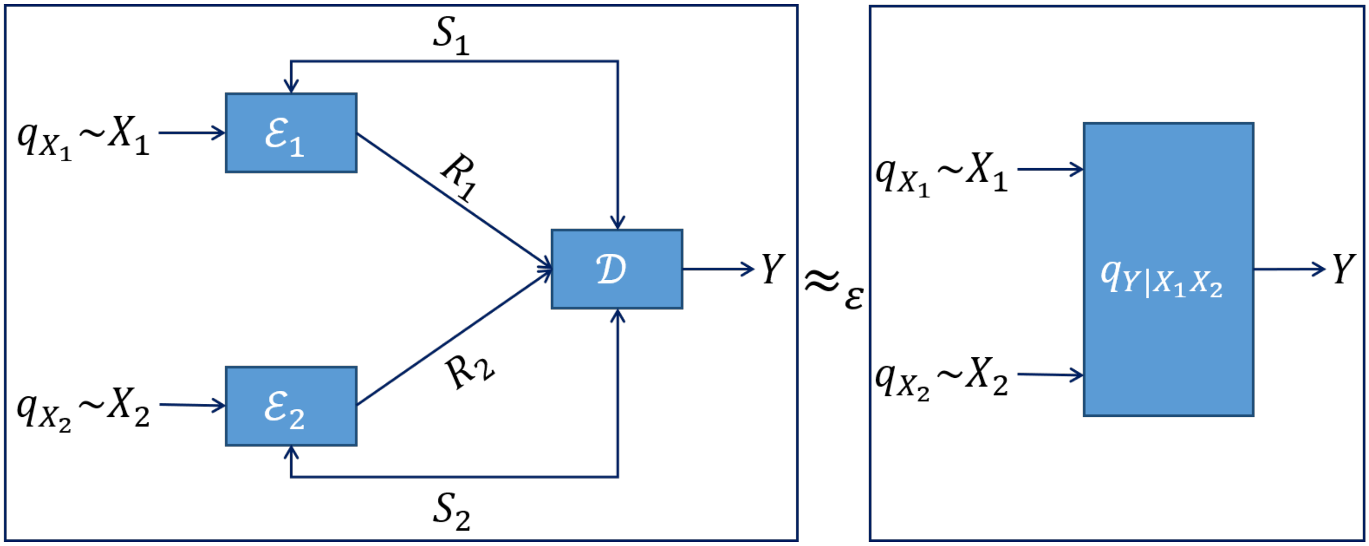

Fig. 1. MAC Simulation: Encoders $\mathcal{E}_j : (X_j, S_j) \xrightarrow[sampling]{rejection} M_j$, Decoder $\mathcal{D} : (M_1, M_2, S_1, S_2) \mapsto Y$; $\approx_\varepsilon$ denotes closeness in tvd.

Similarly, let $\mathcal{R}_{outer}(\varepsilon_1, \varepsilon_2)$ be the set of non-negative real numbers defined as:

$$\mathcal{R}_{outer}(\varepsilon_1, \varepsilon_2) = cl \left\{ \bigcup_{(p_{U_1|X_1}, p_{U_2|X_2}) \in \mathcal{A}_\varepsilon^{outer}} \left\{ (R_1, R_2) : R_j \geq I_{\max}^{\varepsilon_j}(X_j; U_j)_{q_{X_j} p_{U_j|X_j}} \text{ for } j \in \{1, 2\} \right\} \right\}, \tag{8}$$

where

$$\mathcal{A}_\varepsilon^{outer} := \left\{ \left(p_{U_1|X_1}, p_{U_2|X_2}\right) : \exists\, p_{Y|U_1,U_2} \text{ satisfying } \|p_{X_1,X_2,Y} - q_{X_1,X_2,Y}\|_{tvd} \leq 2\varepsilon, |\mathcal{U}_1|, |\mathcal{U}_2| \leq |\mathcal{X}_1||\mathcal{X}_2||\mathcal{Y}| \right\}. \tag{9}$$

The following characterization is our main result in this section.

Theorem 1: Let $q_{Y|X_1,X_2}$ be a given 2-sender, 1-receiver MAC with input $q_{X_1} \times q_{X_2}$. For any $\varepsilon \in (0, 1)$, and $\varepsilon_1, \varepsilon_2, \delta \in (0, 1)$ be such that $\delta < \min\{\varepsilon_1, \varepsilon_2\})$ and $\varepsilon_1 + \varepsilon_2 \leq \varepsilon$, one-shot rate region for simulation of MAC satisfies:

$$\mathcal{R}_{inner}(\varepsilon_1, \varepsilon_2, \delta) \subseteq \mathcal{R}(\varepsilon) \subseteq \mathcal{R}_{outer}(\varepsilon_1, \varepsilon_2), \tag{10}$$

where the inner $(\mathcal{R}_{\mathbb{U}}^{inner}(\varepsilon_1, \varepsilon_2, \delta))$ and the outer $(\mathcal{R}_{\mathbb{U}}^{outer}(\varepsilon_1, \varepsilon_2))$ bounds are as defined in Definition 6.

The proof comprises of two parts, direct part or achievability as shown in Lemma 1.1 and converse as proven in Lemma 1.2.

Remark 1.1: Note that $\mathcal{R}_{inner}(\varepsilon_1, \varepsilon_2, \delta)$ and $\mathcal{R}_{outer}(\varepsilon_1, \varepsilon_2)$ characterize the one-shot rate region $\mathcal{R}(\varepsilon)$ (for $\varepsilon = \varepsilon_1 + \varepsilon_2$) up to a fudge factor that depends on δ. Also, the characterization of $\mathcal{R}(\varepsilon)$ in terms of $\mathcal{R}_{inner}(\varepsilon_1, \varepsilon_2, \delta)$ and $\mathcal{R}_{outer}(\varepsilon_1, \varepsilon_2)$ in Theorem 1 does not involve any sum-rate constraints. This is due to the availability of infinite shared randomness between both the sender-receiver pairs. The output Y plays a role in this characterization through the Markov chain $(X_1, X_2) \to (U_1, U_2) \to Y$ under the distribution $p_{X_1,X_2,U_1,U_2,Y}$.

B. Achievability

Lemma 1.1: For any given $\varepsilon > 0$, let $\varepsilon_1, \varepsilon_2 > 0$ be such that $\varepsilon_1 + \varepsilon_2 \leq \varepsilon$ and $\delta \in (0, \min\{\varepsilon_1, \varepsilon_2\})$. Then, $\mathcal{R}_{inner}(\varepsilon_1, \varepsilon_2, \delta) \subseteq \mathcal{R}(\varepsilon)$.

Proof: Fix $(\varepsilon_1, \varepsilon_2, \delta)$ satisfying the conditions in the lemma and let $q_{X_1} \times q_{X_2}$ be the fixed input distribution. We need to show that for any $(R_1, R_2) \in \mathcal{R}_{inner}(\varepsilon_1, \varepsilon_2, \delta)$ (defined in (6)), there exists an (R_1, R_2, ε) one-shot MAC simulation protocol as mentioned in Definition 5.

Idea: We will use the point-to-point channel simulation algorithm of Fact 2-(i) independently at the two senders.

- **Sender-**j: Let s_{U_j} be a distribution with full support and choose $U_j \sim s_{U_j}$ as the shared randomness between the pair $(\mathcal{E}_j, \mathcal{D})$. Using the rejection sampling algorithm stated in Fact 1, sender j sends the appropriately chosen index of the shared randomness using R_j bits to perform point-to-point channel simulation for the auxiliary channel $p_{U_j|X_j}$.

- **Decoding:** After receiving the transmitted index of shared randomness from both the encoders, the decoder first generates $\{U_j\}_{j=1}^2$ and applies the stochastic map $p_{Y|U_1,U_2}$ to simulate $q_{Y|X_1,X_2}$.

- The output distribution of U_j at $\mathcal{D}$ is denoted by $p^{algo}_{U_j|X_j}$ and satisfies (from Fact 2-(i)) :

$$\left\|p_{U_j,X_j} - q_{U_j,X_j}\right\|_{tvd} = \mathop{\mathbb{E}}_{q_{X_j}} \left\|p^{algo}_{U_j|X_j} - p_{U_j|X_j}\right\|_{tvd} \leq \varepsilon_j. \tag{11}$$

  The amount of classical communication required for this task is given by (see Fact 2-(i)):

$$R_j \geq I^{\varepsilon_j - \delta}_{\max}(X_j; U_j)_p + \log\log\frac{1}{\delta}.$$

Thus, our algorithm results in the overall distribution

$$p^{algo}_{X_1,X_2,U_1,U_2,Y} = q_{X_1} \times q_{X_2} \times p^{algo}_{U_1|X_1} \times p^{algo}_{U_2|X_2} p_{Y|U_1,U_2} \ . \tag{12}$$

To complete the proof, we need to show

$$\mathop{\mathbb{E}}_{q_{X_1} \times q_{X_2}} \|p^{algo}_{Y|X_1,X_2} - q_{Y|X_1,X_2}\|_{tvd} \leq \varepsilon_1 + \varepsilon_2.$$

This follows by the following chain of inequalities:

$$\begin{aligned}
&\mathop{\mathbb{E}}_{q_{X_1} \times q_{X_2}} \|p^{\text{algo}}_{Y|X_1,X_2} - q_{Y|X_1,X_2}\|_{tvd} \\
&\overset{(a)}{\leq} \mathop{\mathbb{E}}_{q_{X_1} \times q_{X_2}} \|p^{\text{algo}}_{Y|X_1,X_2} - p_{Y|X_1,X_2}\|_{tvd} + \mathop{\mathbb{E}}_{q_{X_1} \times q_{X_2}} \|p_{Y|X_1,X_2} - q_{Y|X_1,X_2}\|_{tvd} \\
&\overset{(b)}{=} \mathop{\mathbb{E}}_{q_{X_1} \times q_{X_2}} \left\| \sum_{u_1,u_2} p_{Y|U_1=u_1,U_2=u_2} \left(p^{\text{algo}}_{U_1|X_1}(u_1) p^{\text{algo}}_{U_2|X_2}(u_2) - p_{U_1|X_1}(u_1) p_{U_2|X_2}(u_2) \right) \right\|_{tvd} \\
&= \mathop{\mathbb{E}}_{q_{X_1} \times q_{X_2}} \sum_{u_1,u_2} \sum_{y} p_{Y|U_1,U_2}(y|u_1,u_2) \left| \left(p^{\text{algo}}_{U_1|X_1}(u_1) p^{\text{algo}}_{U_2|X_2}(u_2) - p_{U_1|X_1}(u_1) p_{U_2|X_2}(u_2) \right) \right| \\
&\overset{(c)}{\leq} \mathop{\mathbb{E}}_{q_{X_1} \times q_{X_2}} \|p^{\text{algo}}_{U_1|X_1} p^{algo}_{U_2|X_2} - p^{algo}_{U_1|X_1} p_{U_2|X_2}\|_{tvd} + \mathop{\mathbb{E}}_{q_{X_1} \times q_{X_2}} \|p^{\text{algo}}_{U_1|X_1} p_{U_2|X_2} - p_{U_1|X_1} p_{U_2|X_2}\|_{tvd} \\
&= \mathop{\mathbb{E}}_{q_{X_1}} \|p^{\text{algo}}_{U_1|X_1}\|_1 \mathop{\mathbb{E}}_{q_{X_2}} \|p^{algo}_{U_2|X_2} - p_{U_2|X_2}\|_{tvd} + \mathop{\mathbb{E}}_{q_{X_2}} \|p_{U_2|X_2}\|_1 \mathop{\mathbb{E}}_{q_{X_1}} \|p^{\text{algo}}_{U_1|X_1} - p_{U_1|X_1}\|_{tvd} \\
&\overset{(d)}{\leq} \varepsilon_1 + \varepsilon_2 \ ,
\end{aligned}$$

where (a) and (c) follow from triangle inequality; (b) follows from the definition of distribution induced by the code in (12); and (d) follows from (11). Thus, we have shown that $\mathcal{R}_{inner}(\varepsilon_1, \varepsilon_2) \subseteq \mathcal{R}(\varepsilon)$. ■

C. Converse

Lemma 1.2: For any given $\varepsilon \in (0, 1)$, let $\varepsilon_1, \varepsilon_2 > 0$ be such that $\varepsilon = \varepsilon_1 + \varepsilon_2$. Then, $\mathcal{R}(\varepsilon) \subseteq \mathcal{R}_{outer}(\varepsilon_1, \varepsilon_2)$.

Proof: Let $(\varepsilon_1, \varepsilon_2)$ and ε satisfy the conditions of the lemma. We need to show that any (R_1, R_2, ε) MAC simulation protocol according to Definition 5 has $(R_1, R_2) \in \mathcal{R}_{outer}(\varepsilon_1, \varepsilon_2)$ (defined in (8)).

Consider a MAC simulation protocol with the overall distribution as $\bigotimes_{j=1}^{2}\left(q_{X_j}q_{S_j}p'_{M_j|S_j,X_j}\right)p'_{Y|\vec{M},\vec{S}}$. The encoders are specified by ${p'}_{M_1|X_1,S_1}$ and ${p'}_{M_2|X_2,S_2}$, and the decoder is specified by ${p'}_{Y|M_1,M_2,S_1,S_2}$. Since, the code is a faithful simulation code, we have from Definition 5:

$$\left\|p'_{X_1,X_2,Y}-q_{X_1,X_2,Y}\right\|_{tvd}=\mathop{\mathbb{E}}_{q_{X_1}\times q_{X_2}}\left\|{p'}_{Y|X_1,X_2}-q_{Y|X_1,X_2}\right\|_{tvd}\leq\varepsilon=\varepsilon_1+\varepsilon_2. \tag{13}$$

One of the difficulties in using the standard converse based on information non-locking property of $I_{\max}$ (see e.g. [11, Theorem 5]) is the identification of the auxiliary random variables (U_1,U_2) that are essential for characterizing $\mathcal{R}(\varepsilon)$. Hence, we give a proof inspired from the information spectrum approach (e.g. [12], [13]). The main element of the proof is eliminating a small probability subset of the message and shared randomness for every input symbol. Then, we identify the auxiliary U_j (for $j\in\{1,2\}$) as the tuple of message and the shared randomness restricted to the complement of the above eliminated set. The same intuition applies to the proof of outer bounds of Lemmas 2.2 and 3.2. The formal description now follows.
We now define the following set for every 2-tuple $\vec{x}=(x_1,x_2)$

$$\bar{\mathcal{C}}_{\vec{x}}:=\left\{(\vec{m},\vec{s}):p'_{M_j|S_j,X_j}(m_j|s_j,x_j)\geq\frac{\varepsilon_j}{|\mathcal{M}_j|},j=1,2\right\}. \tag{14}$$

We henceforth denote the projection of $\mathcal{C}_{\vec{x}}$ onto (M_j,S_j,X_j) (or the j^{th} user) as $\mathcal{C}_{x_j}$ and we make the similar identification for their respective complements.
Note that by union bound, we have

$$\mathbb{P}_{p'}(\mathcal{C}_{\vec{x}})\leq\sum_{j=1}^{2}\mathbb{P}\left(\left\{p'_{M_j|S_j,X_j}(m_j|s_j,x_j)\leq\frac{\varepsilon_j}{|\mathcal{M}_j|}\right\}\right)\leq\varepsilon_1+\varepsilon_2, \tag{15}$$

where we have used:

$$\begin{aligned}\mathbb{P}_{p'}(\mathcal{C}_{x_j})&:=\mathbb{P}_{p'}\left(\left\{(m_j,s_j):p'_{M_j|S_j,X_j}(m_j|s_j,x_j)\leq\frac{\varepsilon}{|\mathcal{M}_j|}\right\}\right)\\&=\sum_{(m_j,s_j):p'_{M_j|S_j,X_j}(m_j|s_j,x_j)\leq\frac{\varepsilon_j}{|\mathcal{M}_j|}}p'_{S_j}(s_j)p'_{M_j|S_j,X_j}(m_j|s_j,x_j)\\&\leq\sum_{(m_j,s_j)}\frac{\varepsilon_j}{|\mathcal{M}_j|}q_{S_j}\leq\varepsilon_j.\end{aligned} \tag{16}$$

Hence, $\mathbb{P}_{p'}(\bar{\mathcal{C}}_{\vec{x}})\geq1-\varepsilon_1-\varepsilon_2$, for all $\vec{x}$.
Consider the distribution defined as follows:

$$p_{M_j,S_j|X_j}(m_j,s_j|x_j):=\begin{cases}\frac{p'_{S_j}(s_j)p'_{M_j|S_j,X_j}(m_j|s_j,x_j)}{\mathbb{P}_{p'}(\bar{\mathcal{C}}_{x_j})}, & \text{if } m_j,s_j\in\bar{\mathcal{C}}_{x_j}\\0 & otherwise\ .\end{cases} \tag{17}$$

We have thus identified the auxiliary random variable $\{U_j\}_{j=1}^{2}$ for each x_j as:

$$U_j:=(M_j,S_j)\mathbb{1}_{\bar{\mathcal{C}}_{x_j}}\equiv p_{U_j|X_j}(u_j|x_j):=p_{M_j,S_j|X_j}(m_j,s_j|x_j)=\frac{p'_{S_j}(s_j)p'_{M_j|S_j,X_j}(m_j|s_j,x_j)\mathbb{1}_{(m_j,s_j)\in\bar{\mathcal{C}}_{x_j}}}{\mathbb{P}_{p'}(\bar{\mathcal{C}}_{x_j})}\ .$$

Using this we identify the conditional distribution $p_{\vec{U},Y|\vec{X}}$ (for every $\vec{x}$) as:

$$p_{\vec{U},Y|\vec{X}}(\vec{u},y|\vec{x}):=\begin{cases}\bigotimes_{j=1}^{2}\left[\frac{p'_{S_j}(s_j)p'_{M_j|S_j,X_j}(m_j|s_j,x_j)}{\mathbb{P}_{p'}(\bar{\mathcal{C}}_{x_j})}\right]p'_{Y|\vec{S},\vec{M}}(y|\vec{s},\vec{m}), & \text{if } m_j,s_j\in\bar{\mathcal{C}}_{x_j}\\\left(=\bigotimes_{j=1}^{2}\left[\frac{p'_{U_j|X_j}(u_j|x_j)\mathbb{1}_{m_j,s_j\in\bar{\mathcal{C}}_{x_j}}}{\mathbb{P}_{p'}(\bar{\mathcal{C}}_{x_j})}\right]p'_{Y|\vec{U}}(y|\vec{u})\right), & \\0, & otherwise\ .\end{cases} \tag{18}$$

Now, we identify the complete joint distribution p defined as follows:

$$p_{\vec{X},\vec{U},Y}(\vec{x},\vec{u},y) := \begin{cases} \bigotimes_{j=1}^{2}\left[\frac{q_{X_j}(x_j)p'_{U_j|X_j}(u_j|x_j)}{\mathbb{P}_{p'}(\bar{\mathcal{C}}_{x_j})}\right]p'_{Y|U_1,U_2}(y|u_1,u_2), & \text{if } (\vec{u}) \in \bar{\mathcal{C}}_{\vec{x}} \\ 0, & otherwise~. \end{cases} \tag{19}$$

Note that (16) also gives:

$$\begin{aligned}
&\mathop{\mathbb{E}}_{q_{\vec{X}}}\left\|p_{Y|\vec{X}=\vec{x}} - p'_{Y|\vec{X}=\vec{x}}\right\|_{tvd} \\
&= \mathop{\mathbb{E}}_{q_{\vec{X}}}\left\|\sum_{\vec{m},\vec{s}}\left(p_{\vec{M}=\vec{m},\vec{S}=\vec{s}|\vec{X}} - p'_{\vec{M}=\vec{m},\vec{S}=\vec{s}|\vec{X}}\right)p'_{Y|\vec{M}=\vec{m},\vec{S}=\vec{s}}\right\|_{tvd} \\
&\leq \frac{\sum_{\vec{x}} q_{X_1}(x_1)q_{X_2}(x_2)\left[\sum_{(\vec{m},\vec{s})} p(m_1,s_1|x_1)\left|p(m_2,s_2|x_2) - p'(m_2,s_2|x_2)\right|\right]}{2} \\
&\quad + \frac{\sum_{\vec{x}} q_{X_1}(x_1)q_{X_2}(x_2)\left[\sum_{(\vec{m},\vec{s})} p'(m_2,s_2|x_2)\left|p(m_1,s_1|x_1) - p'(m_1,s_1|x_1)\right|\right]}{2} \\
&\leq \sum_{\vec{x}} q_{X_1}(x_1)q_{X_2}(x_2)\sum_{(m_1,s_1)} p(m_1,s_1|x_1)\times \qquad (20)\\
&\quad \frac{\left[\sum_{(m_2,s_2)\in\bar{\mathcal{C}}_{x_2}}\left|p(m_2,s_2|x_2) - p'(m_2,s_2|x_2)\right| + \sum_{(m_2,s_2)\in\mathcal{C}_{x_2}}\left|p(m_2,s_2|x_2) - p'(m_2,s_2|x_2)\right|\right]}{2} \\
&\quad + \sum_{\vec{x}} q_{X_1}(x_1)q_{X_2}(x_2)\sum_{(m_2,s_2)} p'(m_2,s_2|x_2)]\times \qquad (21)\\
&\quad \frac{\left[\sum_{(m_1,s_1)\in\bar{\mathcal{C}}_{x_1}}\left|p(m_1,s_1|x_1) - p'(m_1,s_1|x_1)\right| + \sum_{(m_1,s_1)\in\mathcal{C}_{x_1}}\left|p(m_1,s_1|x_1) - p'(m_1,s_1|x_1)\right|\right]}{2} \\
&= \frac{\sum_{x_2} q_{X_2}(x_2)\left[\sum_{(m_2,s_2)\in\bar{\mathcal{C}}_{x_2}} p'(m_2,s_2|x_2)\left|\frac{1}{\mathbb{P}_{p'}(\bar{\mathcal{C}}_{x_2})} - 1\right| + \sum_{(m_2,s_2)\in\mathcal{C}_{x_2}} p'(m_2,s_2|x_2)\right]}{2} \\
&\quad + \frac{\sum_{x_1} q_{X_1}(x_1)\left[\sum_{(m_1,s_1)\in\bar{\mathcal{C}}_{x_1}} p'(m_1,s_1|x_1)\left|\frac{1}{\mathbb{P}_{p'}(\bar{\mathcal{C}}_{x_1})} - 1\right| + \sum_{(m_1,s_1)\in\mathcal{C}_{x_1}} p'(m_1,s_1|x_1)\right]}{2} \\
&= \frac{2\sum_{x_1} q_{X_1}(x_1)\mathbb{P}_{p'}(\mathcal{C}_{x_1}) + 2\sum_{x_2} q_{X_2}(x_2)\mathbb{P}_{p'}(\mathcal{C}_{x_2})}{2} \\
&\leq \varepsilon_1 + \varepsilon_2~. \qquad (22)
\end{aligned}$$

Finally, we define the following distribution on the random variable $U_j(= (M_j, S_j))$ that will be used to evaluate the quantity $I^{\varepsilon}_{\max}(X_j;U_j)_p$ for $j \in \{1,2\}$:

$$r_{U_j}(u_j) := q_{S_j}(s_j)\frac{1}{|\mathcal{M}_j|} \tag{23}$$

These identifications leads to the following implications on the rate of the protocol:

$$\begin{aligned}
2^{I^{\varepsilon_j}_{\max}(X_j;U_j)_p} &\overset{(a)}{\leq} 2^{D_{\max}(p'_{X_j,U_j}||p'_{X_j}\times r_{U_j})} \\
&= \max_{x_j}\max_{u_j}\frac{p'_{X_j,U_j}(x_j,u_j)}{p'_{X_j}(x_j)r_{U_j}(u_j)}
\end{aligned}$$

$$\overset{(b)}{=} \max_{x_j} \max_{(m_j,s_j)} \frac{q_{S_j}(s_j) p'_{M_j|S_jX_j}(m_j|s_j,x_j)}{q_{S_j}(s_j)/|\mathcal{M}_j|}$$
$$\overset{(c)}{\leq} |\mathcal{M}_j|, \tag{24}$$

where (a) follows from the definition of smoothed $I_{\max}$ in Definition 3 and observing that distribution $p_{U_j|X_j=x_j} = p_{M_j,S_j|X_j=x_j} \in \mathcal{B}^{\varepsilon_j}(p'_{M_j,S_j|X_j=x_j})$ because:

$$\begin{aligned}\mathop{\mathbb{E}}_{q_{X_j}} \left\| p_{U_j|X_j} - p'_{U_j|X_j} \right\|_{tvd} &= \frac{1}{2}\sum_{x_j} q_{X_j}(x_j) \sum_{m_j,s_j} \left| p'_{M_j,S_j|X_j}(m_j,s_j|x_j) - p_{M_j,S_j|X_j}(m_j,s_j|x_j) \right| \\ &= \frac{1}{2}\sum_{x_j} q_{X_j}(x_j) \left[\sum_{m_j,s_j\in\bar{\mathcal{C}}_{x_j}} p'_{M_j,S_j|X_j}(m_j,s_j|x_j) \left(\frac{1}{\mathbb{P}_{p'}(\bar{\mathcal{C}}_{x_j})} - 1 \right) \right. \\ &\qquad \left. + \sum_{m_j,s_j\in\mathcal{C}_{x_j}} p'_{M_j,S_j|X_j}(m_j,s_j|x_j) \right] \\ &= \mathbb{P}_{p'}(\mathcal{C}_{x_j}) \leq \varepsilon_j \text{ (from (16))};\end{aligned}$$

(b) follows from the identification of $U_j = (M_j, S_j)$ for all p' and the Bayes rule and
(c) follows since $p'_{M_j,S_j|X_j}(m_j,s_j|x_j) \leq 1$ and the definition of $r_j(U_j)$.
We thus have from (24), the rate of the code is lower bounded by:

$$R_j = \log|\mathcal{M}_j| \geq I_{\max}^{\varepsilon_j}(X_j;U_j)_p \text{ for } j \in \{1,2\}.$$

From (22) we have that $p_{Y|\vec{X}=\vec{x}} \in \mathcal{B}^{\varepsilon_1+\varepsilon_2}(p'_{Y|\vec{X}=\vec{x}})$. This along with the simulation constraint of (13) yields by the triangle inequality:

$$\|p_{X_1,X_2,Y} - q_{X_1,X_2,Y}\|_{tvd} \leq 2(\varepsilon_1+\varepsilon_2).$$

Hence, we have shown that for any $(R_1, R_2, 2(\varepsilon_1+\varepsilon_2))$-simulation code the rate of the code is bounded below by:

$$R_j \geq I_{\max}^{\varepsilon_j}(X_j;U_j)_p.$$

To complete the proof, we state the bound on the cardinalities of $\mathcal{U}_1$, $\mathcal{U}_2$ as Lemma 1.3 below.

Lemma 1.3: The cardinalities of $\{\mathcal{U}_1,\mathcal{U}_2\}$ for the region $\mathcal{R}_{outer}$ can be upper bounded as:

$$|\mathcal{U}_j| \leq |\mathcal{X}_1||\mathcal{X}_2||\mathcal{Y}|; \quad \text{for } j \in \{1,2\}\ . \tag{25}$$

The proof of Lemma 1.3 is shown in Appendix B. ■

IV. Asymptotic iid expansion

We now evaluate the asymptotic limit of the iid expansion our one-shot simulation result and show that the cost region in this regime is single-letterized. This recovers a special case of [5, Theorem 1 and Theorem 4] with independent inputs and no side information at the decoder.

For the sake of clarity we start by giving the formal definition of an n-letter MAC channel simulation code.

Definition 7 (Classical MAC simulation with fixed input): An $(nR_1, nR_2, \varepsilon)$ simulation protocol for simulating $q^{\otimes n}_{Y|X_1X_2}$ with inputs $q^{\otimes n}_{X_1} \times q^{\otimes n}_{X_2}$ and access to unlimited shared randomness between Sender1 $\overset{S_1}{\leftrightarrow}$ Receiver and Sender2 $\overset{S_2}{\leftrightarrow}$ Receiver, consists of:

- A pair of encoders of form $\mathcal{E}_1^{(n)} \times \mathcal{E}_2^{(n)}$, such that: $\mathcal{E}_j^{(n)} : \mathcal{X}_j^{(n)} \times \mathcal{S}_j^{(n)} \to \mathcal{M}_j := \left[1 : 2^{nR_j}\right]$ for $j \in \{1,2\}$;
- Two independent noiseless rate-limited links of rate R_j, $j \in \{1,2\}$ and;
- A decoder $\mathcal{D}^{(n)} : \mathcal{M}_1 \times \mathcal{S}_1^{(n)} \times \mathcal{M}_2 \times \mathcal{S}_2^{(n)} \to \mathcal{Y}^{(n)}$;

- The overall joint distribution induced by the encoder-decoder pair is given by

$$p_{X_1^n,X_2^n,S_1^n,S_2^n,M_1,M_2,Y^n} = \left[\mathcal{D}^{(n)} \circ \left(\bigtimes_{j=1}^{2} \mathcal{E}_j^{(n)}\right)\right] \left\{\bigtimes_{j=1}^{2} \left(q_{X_j}^{\otimes n} \times p_{S_j}^{(n)}\right)\right\}$$

s.t.

$$\left\| p_{X_1^n,X_2^n,Y^n} - q_{X_1,X_2,Y}^{\otimes n} \right\|_{tvd} \leq \varepsilon \ . \tag{26}$$

The asymptotic iid rate region $\mathcal{R}^{iid}$ for simulating MAC is defined as the closure of the set of all rate pairs (R_1, R_2) as given above satisfying (26) in the limit $n \to \infty$ followed by $\varepsilon \to 0$.

Henceforth, in this section, we consider $p_{X_1,X_2,U_1,U_2,Y}$ to be a p.m.f. of the form given in (5).

Corollary 1.1: [5, Theorem 1 and Theorem 4] The cost region for simulating a MAC channel $q_{Y|X_1,X_2}$ with fixed inputs $q_{X_1} \times q_{X_2}$, using rate limited links of rate (R_1, R_2) and infinite shared randomness between each sender-receiver pair, in the asymptotic iid limit is given by:

$$\mathcal{R}^{iid} = cl \left\{ \bigcup_{\substack{p_{X_1,X_2,U_1,U_2,Y}: \\ (X_1,X_2)\to(U_1,U_2)\to Y; \\ \& \ p_{X_1,X_2,Y}=q_{X_1,X_2,Y}, \\ |\mathcal{U}_1|,|\mathcal{U}_2|\leq|\mathcal{X}_1||\mathcal{X}_2||\mathcal{Y}|}} \left\{ (R_1, R_2) : R_j \geq I(X_j; U_j)_{q_{X_j} p_{U_j|X_j}} \ ; j \in \{1, 2\} \right\} \right\}. \tag{27}$$

Proof: **Asymptotic iid Inner Bound:** The one-shot inner bound can be straight away extended to obtain the optimal asymptotic iid rate region. Let $(R_1, R_2) \in \mathcal{R}^{iid}$ be such that for any $\eta > 0$,

$$R_j \geq I(X_j; U_j)_p + \eta \text{ for some } p_{X_1,X_2,U_1,U_2,Y} = q_{X_1} q_{X_2} p_{U_1|X_1} p_{U_2|X_2} p_{Y|U_1,U_2} : p_{X_1,X_2,Y} = q_{X_1,X_2,Y}. \tag{28}$$

Consider

$$p_{X_1^n,X_2^n,U_1^n,U_2^n,Y^n} = q_{X_1}^{\otimes n} q_{X_2}^{\otimes n} p_{U_1|X_1}^{\otimes n} p_{U_2|X_2}^{\otimes n} p_{Y|U_1,U_2}^{\otimes n}. \tag{29}$$

The AEP for the smoothed max-mutual information (see (100) of Fact 8) yields

$$\lim_{n\to\infty} \frac{1}{n} \left[I_{\max}^{\varepsilon_j-\delta}(X_j^n, U_j^n)_{p^n} + \log\log\left(\frac{1}{\delta}\right) \right] = I(X_j; U_j)_p,$$

which by (28) means that

$$nR_j \geq I_{\max}^{\varepsilon_j-\delta}(X_j^n, U_j^n)_{p^n} + \log\log\frac{1}{\delta}, \tag{30}$$

for all sufficiently large n (depending on η). This implies that $\mathcal{R}^{iid} \subseteq \mathcal{R}_{inner}^{(n)}(\varepsilon_1, \varepsilon_2)$, where

$$\mathcal{R}_{inner}^{(n)}(\varepsilon_1, \varepsilon_2) = \left\{ (R_1, R_2) : nR_j \geq I_{\max}^{\varepsilon_j-\delta}(X_j^n; U_j^n)_p + \log\log\frac{1}{\delta}; \text{ for } j \in \{1, 2\} \right\}. \tag{31}$$

Asymptotic iid Outer Bound: First note that obtaining the asymptotically optimal outer bound is not so straight forward as the n-fold extension of the random variable U need not be iid. So, we prove a *weak converse*. In order to do so, for any $\varepsilon \in (0, 1)$ we first define the following so-called ε-approximate iid region as follows:

$$\begin{aligned}\mathcal{R}^{iid}(\varepsilon) := \big\{ (R_1, R_2) : R_j \geq I(X_j; U_j)_p, \forall\, & p_{X_1,X_2,U_1,U_2,Y} = q_{X_1} q_{X_2} p_{U_1|X_1} p_{U_2|X_2} p_{Y|U_1,U_2} \\ & \text{such that } \left\| p_{X_1,X_2,Y} - q_{X_1,X_2,Y} \right\|_{tvd} \leq \varepsilon \big\}\end{aligned} \tag{32}$$

For any $\varepsilon \in (0, 1)$ and $\varepsilon_1, \varepsilon_2 > 0$ such that $\max\{\varepsilon_1, \varepsilon_2\} \leq \varepsilon/4$, let $\mathcal{R}_{outer}^{(n)}(\varepsilon_1, \varepsilon_2, \varepsilon)$ be the n-fold extension of the region $\mathcal{R}_{outer}(\varepsilon_1, \varepsilon_2, \varepsilon)$ with respect to the input and auxiliary random variables $(X_j^n, U_j^n) \sim q_{X_j}^{\otimes n} p_{U_j^n|X_j^n}$, i.e.

$$\mathcal{R}_{outer}^{(n)}(\varepsilon_1, \varepsilon_2, \varepsilon) = \left\{ (R_1, R_2) : nR_j \geq I_{\max}^{\varepsilon_j}(X_j^n; U_j^n)_p; \text{ for } j \in \{1, 2\} \right\}, \tag{33}$$

where $p_{X_1^n,U_1^n,X_2^n,U_2^n,Y^n} := q_{X_1}^{\otimes n} q_{X_2}^{\otimes n} p_{U_1^n|X_1^n} p_{U_2^n|X_2^n} p_{Y^n|U_1^n,U_2^n}$ is such that

$$\left\| p^n_{X_1,X_2,Y} - q^{\otimes n}_{X_1,X_2,Y} \right\|_{tvd} \leq \varepsilon \text{ (as } 2(\varepsilon_1 + \varepsilon_2) \leq \varepsilon). \tag{34}$$

Suppose $(R_1, R_2) \in \mathcal{R}^{(n)}_{outer}(\varepsilon_1, \varepsilon_2, \varepsilon)$. Then:

$$\begin{aligned}
nR_j &\geq I^{\varepsilon_j}_{\max}(X^n_j; U^n_j)_{p^n} \\
&\overset{(a)}{=} I_{\max}(X^n_j; U^n_j)_{p'^n} \\
&\overset{(b)}{\geq} I(X^n_j; U^n_j)_{p'^n} \\
&\overset{(c)}{\geq} I(X^n_j; U^n_j)_{p^n} - 2\varepsilon_j \log |\mathcal{X}_j|^n - 2h_2\left(\frac{\varepsilon_j}{1+\varepsilon_j}\right) \\
&\overset{(d)}{\geq} nI(X_j; U_j)_p - 2\varepsilon \log |\mathcal{X}_j|^n - 2h_2\left(\frac{\varepsilon_j}{1+\varepsilon_j}\right) \\
&\Rightarrow R_j \geq \lim_{\varepsilon \to 0} \lim_{n \to \infty} \left[\frac{nI(X_j; U_j)_p - 2\varepsilon \log |\mathcal{X}_j|^n - 2h_2\left(\frac{\varepsilon_j}{1+\varepsilon_j}\right)}{n} \right] \\
&\Rightarrow R_j \geq I(X_j; U_j)_p,
\end{aligned}$$

where (a) holds by taking $p'_{X^n_j,U^n_j} \in \mathcal{B}^{\varepsilon_j}(p_{X^n_j,U^n_j})$ to be the optimizer for $I^{\varepsilon}_{\max}$; (b) holds by the fact the $I_{\max}(X;Y)_p \geq I(X;Y)_p$ for any joint distribution $p_{X,Y}$; (c) follows due to continuity of mutual information from Fact 6; (d) follows by Proposition 1 shown in Appendix B-D for some $p_{X_j,U_j} = q_{X_j} p_{U_j|X_j}$ and finite $|\mathcal{U}_j|$ from Lemma 1.3. Note that (34) and monotonicity of trace distance implies that $\|p_{X_1,X_2,Y} - q_{X_1,X_2,Y}\|_{tvd} \leq \delta$. Hence, we have shown that in the asymptotic iid limit:

$$\lim_{\varepsilon \to 0} \lim_{n \to \infty} \mathcal{R}^{(n)}_{outer}(\varepsilon_1, \varepsilon_2, \varepsilon) \subseteq \mathcal{R}^{iid}(\varepsilon). \tag{35}$$

We have thus recovered the asymptotically optimal region of [5, Theorem 1, Theorem 3] up to δ. Since, in our setting we have bounded cardinalities of the auxiliary random variables, we can directly apply [14, Lemma 6] in our case (see Fact 7 for a detailed analysis). We thus finally recover the asymptotically optimal region of [5, Theorem 1, Theorem 3] in our setting of independent and fixed inputs and no side information at the decoder, to get

$$\mathcal{R}_{outer} := \lim_{\varepsilon \to 0} \lim_{n \to \infty} \mathcal{R}^{(n)}_{outer}(\varepsilon_1, \varepsilon_2, \varepsilon) \subseteq \lim_{\varepsilon \to 0} \mathcal{R}^{iid}(\varepsilon) = \mathcal{R}^{iid}.$$

Thus we have shown that

$$\begin{aligned}
\mathcal{R}_{outer} \subseteq \mathcal{R}^{iid} \subseteq \lim_{n \to \infty} \mathcal{R}^{(n)}_{inner}(\varepsilon_1, \varepsilon_2) \subseteq \mathcal{R}_{outer} \\
\Rightarrow \mathcal{R}_{inner} := \lim_{\varepsilon_1, \varepsilon_2 \to 0} \lim_{n \to \infty} \mathcal{R}^{(n)}_{inner}(\varepsilon_1, \varepsilon_2) = \mathcal{R}^{iid} = \mathcal{R}_{outer} .
\end{aligned}$$

■

V. Universal MAC simulation

Here, we consider the task of universal channel simulation, where the protocol should simulate the channel $q_{Y|X_1,X_2}$ irrespective of any particular choice of input distribution $q_{X_1} \times q_{X_2}$. Note the the inputs of the two senders are still independent, but arbitrary. In the next proposition we show that Lemma 1.1 and Lemma 1.2 of our MAC simulation protocol can be extended to achieve universal simulation with appropriate modifications. These modifications refer to the simulation error be replaced by the maximum over the input samples x_1, x_2 (sampled according to any input distribution) in contrast with the average over a fixed input distribution. Before stating our result, we define *universal* protocol for MAC simulation formally.

Definition 8 (Universal MAC simulation): An (R_1, R_2, ε) simulation protocol for a 2-independent user MAC $q_{Y|X_1X_2}$ with inputs $q_{X_1} \times q_{X_2}$ and access to unlimited shared randomness between Sender1$\overset{S_1}{\leftrightarrow}$ Receiver and Sender2$\overset{S_2}{\leftrightarrow}$ Receiver, consists of:

- A pair of encoders of form $\mathcal{E}_1 \times \mathcal{E}_2$, such that: $\mathcal{E}_j : \mathcal{X}_j \times \mathcal{S}_j \to [1 : 2^{R_j}]$, for $j \in \{1, 2\}$;
- Two independent noiseless rate-limited links of rate R_j, $j \in \{1, 2\}$ and;
- A decoder $\mathcal{D} : [1 : 2^{R_1}] \times \mathcal{S}_1 \times [1 : 2^{R_2}] \times \mathcal{S}_2 \to \mathcal{Y}$;
- The overall joint distribution induced by the encoder-decoder pair final output is given by

$$p_{X_1,X_2,S_1,S_2,M_1,M_2,Y} = \left[\mathcal{D} \circ \left(\mathop{\times}_{j=1}^{2} \mathcal{E}_j\right)\right] \left\{\mathop{\times}_{j=1}^{2} \left(q_{X_j} \times p_{S_j}\right)\right\}$$

s.t.

$$\max_{x_1,x_2} \left\|p_{Y|X_1=x_1,X_2=x_2} - q_{Y|X_1=x_1,X_2=x_2}\right\|_{tvd} \leq \varepsilon \text{ and } R_1 = \log|\mathcal{M}_1|,\ R_2 = \log|\mathcal{M}_2|\ . \tag{36}$$

The rate region $\mathcal{R}_{\mathbb{U}}(\varepsilon)$ for universal simulation of a MAC is defined as the closure of the set of all rate pairs (R_1, R_2) as given above satisfying (36).

We say that the simulation protocol of Definition 8 is *universal* in the sense that it can simulate the given MAC $q_{Y|X_1,X_2}$ for any input distribution $q_{X_1} \times q_{X_2}$, without being dependent on q_{X_1} and q_{X_2}. This is ensured by the $\max$-error criterion in (36).

In this section, we henceforth consider $p_{U_1,U_2,Y|X_1,X_2}$ to be a conditional p.m.f. of the form:

$$p_{U_1,U_2,Y|X_1,X_2} = p_{U_1|X_1} p_{U_2|X_2} p_{Y|U_1,U_2}. \tag{37}$$

A. One-shot setting

We first introduce the regions $\mathcal{R}_{\mathbb{U}}^{inner}$ and $\mathcal{R}_{\mathbb{U}}^{outer}$ which we will prove are the respective inner and outer bounds for the task of one-shot universal MAC simulation given by Definition 8.

Definition 9: Let $\varepsilon \in (0, 1)$, and $\varepsilon_1, \varepsilon_2, \delta \in (0, 1)$ be such that $\delta < \min\{\varepsilon_1, \varepsilon_2\}$ and $\varepsilon_1 + \varepsilon_2 \leq \varepsilon$. Let $\mathcal{R}_{\mathbb{U}}^{inner}(\varepsilon_1, \varepsilon_2, \delta)$ be the set of non-negative real numbers (R_1, R_2) defined as:

$$\mathcal{R}_{\mathbb{U}}^{inner}(\varepsilon_1, \varepsilon_2, \delta) = cl \left\{ \bigcup_{\left(p_{U_1|X_1}, p_{U_2|X_2}\right) \in \mathcal{A}^{inner}} \left\{(R_1, R_2) : R_j \geq I_{\max}^{\varepsilon_j - \delta}(p_{U_j|X_j}) + \log\log\frac{1}{\delta} \text{ for } j \in \{1, 2\}\right\} \right\}, \tag{38}$$

where $\mathcal{A}^{inner}$ is the set of all feasible distributions for evaluating $\mathcal{R}_{\mathbb{U}}^{inner}$ and is given by

$$\mathcal{A}^{inner} := \left\{\left(p_{U_1|X_1}, p_{U_2|X_2}\right) :\ \exists\ p_{Y|U_1,U_2} \text{ satisfying } p_{Y|X_1,X_2} = q_{Y|X_1,X_2}\right\}. \tag{39}$$

Similarly, let $\mathcal{R}_{\mathbb{U}}^{outer}(\varepsilon_1, \varepsilon_2)$ be the set of non-negative real numbers (R_1, R_2) defined as:

$$\mathcal{R}_{\mathbb{U}}^{outer}(\varepsilon_1, \varepsilon_2) = cl \left\{ \bigcup_{\left(p_{U_1|X_1}, p_{U_2|X_2}\right) \in \mathcal{A}_{\varepsilon}^{outer}} \left\{(R_1, R_2) : R_j \geq I_{\max}^{\varepsilon_j}(p_{U_j|X_j}) \text{ for } j \in \{1, 2\}\right\} \right\}, \tag{40}$$

where $\mathcal{A}_{\varepsilon}^{outer}$ is the set of all feasible distributions for evaluating $\mathcal{R}_{\mathbb{U}}^{outer}$ and is given by

$$\mathcal{A}_{\varepsilon}^{outer} := \left\{ \left(p_{U_1|X_1}, p_{U_2|X_2}\right) :\ \exists\ p_{Y|U_1,U_2} \text{ satisfying} \right. \tag{41}$$

$$\left. \max_{x_1,x_2} \left\|p_{Y|X_1=x_1,X_2=x_2} - q_{Y|X_1=x_1,X_2=x_2}\right\|_{tvd} \leq 2\varepsilon,\ |\mathcal{U}_1|, |\mathcal{U}_2| \leq |\mathcal{X}_1||\mathcal{X}_2||\mathcal{Y}| \right\}. \tag{42}$$

Theorem 2: Let $q_{Y|X_1,X_2}$ be a given 2-sender, 1-receiver MAC. For any $\varepsilon \in (0,1)$, and $\varepsilon_1, \varepsilon_2, \delta \in (0,1)$ be such that $\delta < \min\{\varepsilon_1, \varepsilon_2\})$ and $\varepsilon_1 + \varepsilon_2 \leq \varepsilon$, one-shot rate region for *universal* simulation of MAC satisfies:

$$\mathcal{R}_{\mathbb{U}}^{inner}(\varepsilon_1, \varepsilon_2, \delta) \subseteq \mathcal{R}_{\mathbb{U}}(\varepsilon) \subseteq \mathcal{R}_{\mathbb{U}}^{outer}(\varepsilon_1, \varepsilon_2) \ , \tag{43}$$

where $\mathcal{R}_{\mathbb{U}}^{inner}(\varepsilon_1, \varepsilon_2, \delta)$ and $\mathcal{R}_{\mathbb{U}}^{outer}(\varepsilon_1, \varepsilon_2)$ are as defined in Definition 9.

The proof comprises of two parts:

- Direct part or the achievability as shown in Lemma 2.1; and
- Converse as proven in Lemma 2.2.

B. Achievability

Lemma 2.1: For any given $\varepsilon > 0$, let $\varepsilon_1, \varepsilon_2 > 0$ be such that $\varepsilon_1 + \varepsilon_2 \leq \varepsilon$ and $\delta \in (0, \min\{\varepsilon_1, \varepsilon_2\})$. Then, $\mathcal{R}_{\mathbb{U}}^{inner}(\varepsilon_1, \varepsilon_2, \delta) \subseteq \mathcal{R}_{\mathbb{U}}(\varepsilon)$.

The proof of this lemma (see Appendix D-A) is very similar to that of the fixed input simulation case of Lemma 1.1, with the difference being that the simulation error criterion is changed from average to maximum.

C. Converse

Lemma 2.2: For any given $\varepsilon \in (0,1)$, let $\varepsilon_1, \varepsilon_2 > 0$ be such that $\varepsilon = \varepsilon_1 + \varepsilon_2$. Then, $\mathcal{R}_{\mathbb{U}}(\varepsilon) \subseteq \mathcal{R}_{\mathbb{U}}^{outer}(\varepsilon_1, \varepsilon_2)$.

The proof has minor technical changes compared to that of Lemma 1.2 due to the average simulation error being replaced by the maximum simulation error criterion and is given in Appendix D-B.

We now extend the one-shot result to the asymptotic iid setting.

D. Asymptotic expansion

In this section, we consider a universal MAC simulation protocol that simulates n-iid copies of the channel, that is, $q_{Y|X_1,X_2}^{\otimes n}$ with general n-letter inputs denoted by $q_{X_1^n} \times q_{X_2^n}$. We note that this is in contrast to the generic usage of the term asymptotic iid, as used in previous sections, which refers to iid inputs $q_{X_1}^{\otimes n} \times q_{X_2}^{\otimes n}$. This leads to non-trivialities in extending the one-shot result to asymptotic iid as neither the inputs nor the auxiliary random variables that characterize the rate region are iid. Nevertheless we prove the following single-letter characterization even for this case.
Henceforth, in this section, we consider $p_{U_1,U_2,Y|X_1,X_2}$ to be a conditional p.m.f. of the form given in (37).

Corollary 2.1: The rate region for *universal* asymptotic iid simulation of MAC $q_{Y|X_1,X_2}$ is given by

$$\mathcal{R}_{\mathbb{U}}^{iid} = cl \left\{ \bigcup_{\substack{p_{U_1,U_2,Y|X_1,X_2}: \\ (X_1,X_2)\to(U_1,U_2)\to Y \\ \& \ p_{Y|X_1,X_2}=q_{Y|X_1,X_2}, \\ |\mathcal{U}_1|,|\mathcal{U}_2|\leq|\mathcal{X}_1||\mathcal{X}_2||\mathcal{Y}|}} \left\{ (R_1, R_2) : R_j \geq \max_{q_{X_j}} I(X_j; U_j)_{q_{X_j} p_{U_j|X_j}}, \quad j \in \{1,2\} \right\} \right\}. \tag{44}$$

We prove this proposition in Appendix D-C.

Remark 2.1: Note that the point-to-point channel is a special case of the MAC, where one of the inputs, say X_2, is redundant. Then, setting $U_1 = Y$, $U_2 = X_2 = 1$ with probability one, and $\varepsilon_2 = 0$ in the definitions of $\mathcal{R}_{inner}(\varepsilon_1, \varepsilon_2, \delta)$ and $\mathcal{R}_{outer}(\varepsilon_1, \varepsilon_2)$, we recover the one-shot point-to-point channel simulation result of [3] stated in Fact 2 (see Appendix A). Furthermore, this also recovers the asymptotically optimal point-point channel simulation rate $I(X;Y)_{q_{X,Y}}$ as shown in [3], [15]. Moreover, for the point-to-point case our technique can be straight away extended to obtain the universal channel simulation by identifying optimal $U = Y$.

Remark 2.2: We remark that the universal simulation protocol can also be used for the fixed input protocol of Section III. The main difference is that the communication rates of the universal protocol are higher than that is necessary for the fixed input case because

$$I^{\varepsilon}_{\max}(p_{U|X}) \geq I^{\varepsilon}_{\max}(X;U)_{q_X p_{U|X}}$$

due to the fact that the minimization in the definition of the channel smoothed max-mutual information (left hand side term above) is over a small set since the simulation error criterion is stronger. Hence, we gave a separate analysis for the fixed input case.

VI. Quantum-classical MAC simulation

In this section, we take a step towards generalizing our simulation protocol in the quantum regime. To this end, we consider a MAC with two independent quantum inputs and one classical output. Generally, a channel that takes a quantum state as an input and outputs a probability distribution (or classical state as the output random variable) is modelled as a measurement device (or a measurement channel). Hence we refer to the 2-quantum input and 1-classical output as a QC MAC and think of it as a measurement channel.

Classical scrambling QC MAC (CS-QC MAC): Channel first does a product measurement on two inputs with classical outcomes X_1, X_2 and then scrambles them according to the conditional probability distribution $q_{Y|X_1,X_2}$. We refer to such channels as "*classical scrambling*" (CS) channels, denoted as

$$\mathcal{N}_{CS}^{A_1A_2\to Y} := q_{Y|X_1,X_2} \circ \left(\Lambda^{A_1\to X_1} \otimes \Gamma^{A_2\to X_2}\right), \tag{45}$$

where $\Lambda^{A_1\to X_1}$ and $\Gamma^{A_2\to X_2}$ are measurements with POVM elements $\{\Lambda_{x_1}\}_{x_1}$ and $\{\Gamma_{x_2}\}_{x_2}$, respectively. This is a special case of the model of a QC-channel proposed in [16], termed as *distributed measurement channel having a separable decomposition with stochastic integration*.

We characterize the cost of simulating the CS-QC MAC with feedback defined as follows:

CS-QC MAC with feedback: Channel first does a product measurement on two inputs, creates two copies of the classical outputs and then scrambles one of the copies according to $q_{Y|X_1,X_2}$, while keeping the other untouched. We refer to such channels as "*classical scrambling channels with feedback*", shown in the right hand side of Figure VI and is denoted as:

$$\mathcal{N}_{CS}^{A_1A_2\to YX_1X_2} := q_{Y|X_1,X_2} \circ \left(\Lambda^{A_1\to X_1X_1'} \otimes \Gamma^{A_2\to X_2X_2'}\right),$$

where the measurement operators are defined as:

$$\mathcal{I}^{E_1} \otimes \Lambda^{A_1\to X_1X_1'}(|\varphi_1\rangle\langle\varphi_1|^{E_1A_1}) := \sum_{x_1} p_{X_1}(x_1)\, |x_1\rangle\langle x_1|^{X_1} \otimes |x_1\rangle\langle x_1|^{X_1'} \otimes \varphi_{x_1}^{E_1} \ : \mathrm{Tr}(\varphi_{x_1}^{E_1}) = 1;$$
$$\mathcal{I}^{E_2} \otimes \Gamma^{A_2\to X_2X_2'}(|\varphi_1\rangle\langle\varphi_1|^{E_2A_2}) := \sum_{x_2} p_{X_2}(x_2)\, |x_2\rangle\langle x_2|^{X_2} \otimes |x_2\rangle\langle x_2|^{X_2'} \otimes \varphi_{x_2}^{E_2} \ : \mathrm{Tr}(\varphi_{x_2}^{E_2}) = 1.$$

Note that (X_1', X_2') are just the classical copies of (X_1, X_2). The conditional distribution $q_{Y|X_1,X_2}$ is a probability measure on $\mathcal{Y}$ conditioned on random variables taking values in $\mathcal{X}_1 \times \mathcal{X}_2$. Henceforth, we consistently use the notation $q_{Y|X_1,X_2}$ (instead of $q_{Y|X_1',X_2'}$) to represent the classical scrambling map to mean that random variables (X_1', X_2') are stochastically mapped to the output random variable Y. Thus, the actual channel outcome is given as

$$\mathcal{N}_{CS}^{AB\to YX_1X_2}(\rho_1^{A_1} \otimes \rho_2^{A_2}) = \sum_{x_1,x_2,y} q_{Y|X_1,X_2}(y|x_1,x_2)\, |y\rangle\langle y|^Y \otimes \mathrm{Tr}_{A_1}[\Lambda_{x_1}^{A_1}\rho_1]\, |x_1\rangle\langle x_1|^{X_1} \otimes \mathrm{Tr}_{A_2}[\Gamma_{x_2}^{A_2}\rho_2]\, |x_2\rangle\langle x_2|^{X_2}. \tag{46}$$

Here, we focus on the task of simulating CS-QC MAC with feedback represented by (46).

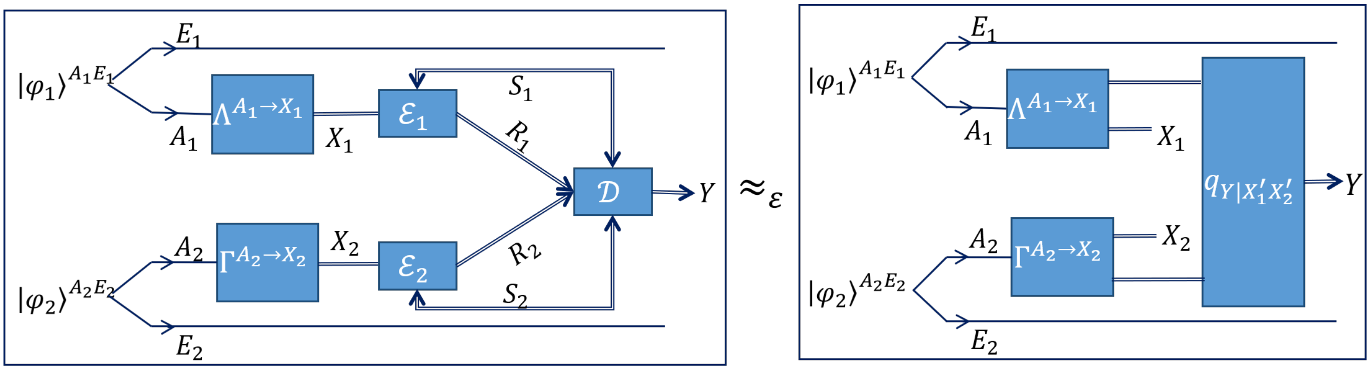

Fig. 2. CS-QC MAC with feedback: Encoders $\mathcal{E}_j : (X_j', S_j) \xrightarrow[split]{convex} M_j \in [1:2^{R_j}]$, Decoder $\mathcal{D} : (M_1, M_2, S_1, S_2) \mapsto Y$; $\approx_\varepsilon$ denotes closeness in tvd.

A. Feedback simulation for fixed product input

Definition 10 (CS-QC MAC with feedback simulation): An (R_1, R_2, ε) simulation code for a 2-independent user CS-QC MAC with feedback given in (46) and access to unlimited shared randomness between Sender1 $\overset{S_1}{\leftrightarrow}$ Receiver and Sender2 $\overset{S_2}{\leftrightarrow}$ Receiver, consists of:

- Inputs to the two encoders are ${\varphi_1}^{E_1X_1X_1'} \otimes {\varphi_2}^{E_2X_2X_2'}$, where
$$ {\varphi_j}^{E_jX_jX_j'} := \sum_{x_j} p_{X_j(x_j)} \left|x_j\middle\rangle\!\middle\langle x_j\right|^{X_j} \otimes \left|x_j\middle\rangle\!\middle\langle x_j\right|^{X_j'} \otimes {\varphi_{x_j}}^{E_j};\ p_{X_j}(x_j) := \mathrm{Tr}[\Lambda_{x_j}\rho_j], $$
and $\varphi_{x_1}^{E_1} \otimes \varphi_{x_2}^{E_2}$ is the normalized post-measurement state of the measurement $\Lambda \otimes \Gamma$. Note that X' is just a classical copy of X, to perform the simulation with feedback;
- A pair of encoders $\mathcal{E}_1 \otimes \mathcal{E}_2$ with inputs as the measurement outcomes X_1, X_2 and shared randomness S_1 S_2, denoted by: $\mathcal{E}_j : \mathcal{X}_j \otimes \mathcal{S}_j \to [1:2^{R_j}]$, for $j \in \{1,2\}$;
- Two separate noiseless rate-limited classical links of rate R_j, $j \in \{1,2\}$ and;
- A decoder $\mathcal{D} : [1:2^{R_1}] \times \mathcal{S}_1 \times [1:2^{R_2}] \times \mathcal{S}_2 \to \mathcal{Y}$;
- The simulation algorithm produces the overall state as
$$ \tau^{YX_1X_2E_1E_2} := \mathcal{D} \circ \left(\bigotimes_{j=1}^{2} \mathcal{E}_j \right) \left(|\varphi_1\rangle\!\langle\varphi_1| \otimes |\varphi_2\rangle\!\langle\varphi_2| \otimes \left(\bigotimes_{j=1}^{2} S_j \right) \right), $$
such that
$$ \left\| \tau^{YX_1X_2E_1E_2} - \eta^{YX_1X_2E_1E_2} \right\|_1 \leq \varepsilon, \tag{47} $$
where $\eta^{YX_1X_2E_1E_2} := \mathcal{N}^{A_1A_2\to YX_1X_2}(|\varphi_1\rangle\!\langle\varphi_1|^{A_1E_1} \otimes |\varphi_2\rangle\!\langle\varphi_2|^{A_2E_2})$, is the output state of CS-QC MAC $\mathcal{N}$ to be simulated.

The rate region $\mathcal{R}(\varepsilon)$ for simulating MAC is defined as the closure of the set of all rate pairs (R_1, R_2) as given above.

The classical MAC simulation described in Section III is the non-feedback simulation. However, in order to extend the classical proof technique of the converse to CS-QC MAC, we require the encoders to have access to the classical outcomes of the measurement channels $\Lambda^{A_1\to X_1} \otimes \Gamma^{A_2\to X_2}$. Observe that simulation criteria in (47) is an average error criterion similar to the classical MAC simulation criteria with fixed inputs given in (4).
In what follows, let $\tau^{E_1E_2X_1X_2U_1U_2Y}$ and $\eta^{E_1E_2X_1X_2U_1U_2Y}$ be the following classical-quantum (CQ) states:

$$ \begin{aligned} \tau^{E_1E_2U_1U_2X_1X_2Y} \quad := \quad & \sum_{\vec{u},\vec{x},y} p_{Y|U_1U_2}(y|u_1,u_2) p_{U_1|X_1}(u_1|x_1) p_{X_1}(x_1) p_{U_2|X_2}(u_2|x_2) p_{X_2}(x_2) \left|y\middle\rangle\!\middle\langle y\right|^{Y} \otimes \left|x_1\middle\rangle\!\middle\langle x_1\right|^{X_1} \\ & \otimes \left|u_1\middle\rangle\!\middle\langle u_1\right|^{U_1} \otimes \left|x_2\middle\rangle\!\middle\langle x_2\right|^{X_2} \left|u_2\middle\rangle\!\middle\langle u_2\right|^{U_2} \otimes \frac{\mathrm{Tr}_{A_1}\left[\left\{ \mathcal{I}^{E_1} \otimes \Lambda_{x_1}^{A} \right\} (\varphi_1^{E_1A_1}) \right]}{p_{X_1}(x_1)} \otimes \end{aligned} \tag{48} $$

$$\frac{\mathrm{Tr}_{A_2}\left[\left\{\mathcal{I}^{E_2}\otimes\Gamma^{A_2}_{x_2}\right\}(\varphi_2^{E_2A_2})\right]}{p_{X_2}(x_2)}; \qquad \text{and} \tag{49}$$

$$\eta^{YE_1E_2X_1X_2} := \sum_{\vec{x},y} q_{Y|X_1,X_2}(y|x_1,x_2)p_{X_1}(x_1)\otimes p_{X_2}(x_2)\, |y\rangle\langle y|^{Y}\otimes|x_1\rangle\langle x_1|^{X_1}\otimes|x_2\rangle\langle x_2|^{X_2}\otimes \frac{\mathrm{Tr}_{A_1}\left[\left\{\mathcal{I}^{E_1}\otimes\Lambda^{A}_{x_1}\right\}(\varphi_1^{E_1A_1})\right]}{p_{X_1}(x_1)}\otimes\frac{\mathrm{Tr}_{A_2}\left[\left\{\mathcal{I}^{E_2}\otimes\Gamma^{A_2}_{x_2}\right\}(\varphi_2^{E_2A_2})\right]}{p_{X_2}(x_2)}. \tag{50}$$

We now define the regions analogous to the classical case given by Definition 6 and 9, which will be proven as the *inner and outer bound* regions for the task of QC-MAC simulation *with feedback* analogous to Definition 6.

Definition 11 (Inner and Outer bounds): Let $\varepsilon\in(0,1)$, and $\varepsilon_1,\varepsilon_2,\delta\in(0,1)$ be such that $\delta<\min\{\varepsilon_1,\varepsilon_2\}$ and $\varepsilon_1+\varepsilon_2\leq\varepsilon$. Let $\mathcal{R}^{QC-fb}_{inner}(\varepsilon_1,\varepsilon_2,\delta)$ be the set of non-negative real numbers (R_1,R_2) defined as:

$$\mathcal{R}^{QC-fb}_{inner}(\varepsilon_1,\varepsilon_2,\delta)=cl\left\{\bigcup_{(\tau^{E_1X_1U_1},\tau^{E_2X_2U_2})\in\mathcal{A}^{inner}}\left\{(R_1,R_2): R_j\geq I^{\varepsilon_j-\delta}_{\max}(E_j;U_j)_\tau+2\log\frac{1}{\delta}\ \text{ for } j\in\{1,2\}\right\}\right\}, \tag{51}$$

where $\mathcal{A}^{inner}$ is the set of all feasible states for evaluating $\mathcal{R}^{QC-fb}_{inner}$ and is given by

$$\mathcal{A}^{inner}:=\left\{\left(\tau^{E_1X_1U_1},\tau^{E_2X_2U_2}\right):\ \forall\tau \text{ s.t. } \tau^{E_1E_2X_1X_2Y}=\eta^{E_1E_2X_1X_2Y}\right\}. \tag{52}$$

Similarly, let $\mathcal{R}^{QC-fb}_{outer}(\varepsilon_1,\varepsilon_2)$ be the set of non-negative real (R_1,R_2) numbers defined as:

$$\mathcal{R}^{QC-fb}_{outer}(\varepsilon_1,\varepsilon_2)=cl\left\{\bigcup_{(\tau^{E_1X_1U_1},\tau^{E_2X_2U_2})\in\mathcal{A}^{outer}_{\varepsilon}}\{(R_1,R_2): R_j\geq I^{\varepsilon_j}_{\max}(E_j;U_j)_\tau \text{ for } j\in\{1,2\}\}\right\}, \tag{53}$$

where $\mathcal{A}^{outer}_{\varepsilon}$ is the set of all feasible states for evaluating $\mathcal{R}^{QC-fb}_{outer}$ and is given by

$$\mathcal{A}^{outer}_{\varepsilon}:=\left\{\left(\tau^{E_1X_1U_1},\tau^{E_2X_2U_2}\right):\ \forall\tau \text{ s.t. } \left\|\tau^{E_1E_2X_1X_2Y}-\eta^{E_1E_2X_1X_2Y}\right\|_{tvd}\leq 2\varepsilon, |\mathcal{U}_1|,|\mathcal{U}_2|\leq|\mathcal{X}_1||\mathcal{X}_2||\mathcal{Y}|\right\}. \tag{54}$$

The following characterization is our main result in this section.

Theorem 3: Let $\mathcal{N}^{AB\to YX_1X_2}_{CS}$ be a given 2-sender, 1-receiver MAC with input $\rho^{A_1}\otimes\rho^{A_2}$ and their respective purifications denoted by $|\varphi\rangle^{E_1A_1}\otimes|\varphi\rangle^{E_2A_2}$. For $\varepsilon\in(0,1)$ and $\varepsilon_1,\varepsilon_2>0$ with $\varepsilon_1+\varepsilon_2\leq\varepsilon$, and $\delta\in(0,\min\{\varepsilon_1,\varepsilon_2\})$, we have

$$\mathcal{R}^{QC-fb}_{inner}(\varepsilon_1,\varepsilon_2,\delta)\subseteq\mathcal{R}(\varepsilon)\subseteq\mathcal{R}^{QC-fb}_{outer}(\varepsilon_1,\varepsilon_2), \tag{55}$$

where $\mathcal{R}^{QC-fb}_{inner}(\varepsilon_1,\varepsilon_2,\delta)$ and $\mathcal{R}^{QC-fb}_{outer}(\varepsilon_1,\varepsilon_2)$ are given in Definition 11.

The proof comprises of two parts:

- Direct part or the achievability, which we prove in Lemma 3.1; and
- Converse shown in Lemma 3.2.

Note that $\mathcal{R}^{QC-fb}_{inner}(\varepsilon_1,\varepsilon_2,\delta)$ and $\mathcal{R}^{QC-fb}_{outer}(\varepsilon_1,\varepsilon_2)$ characterize the one-shot rate region $\mathcal{R}(\varepsilon)$ up to a fudge factor that depends on δ,ε_1 and ε_2.

B. Achievability

Lemma 3.1: For any given $\varepsilon>0$, let $\varepsilon_1,\varepsilon_2>0$ be such that $\varepsilon_1+\varepsilon_2\leq\varepsilon$ and $\delta\in(0,\min\{\varepsilon_1,\varepsilon_2\})$. Then $\mathcal{R}^{QC-fb}_{inner}(\varepsilon_1,\varepsilon_2,\delta)\subseteq\mathcal{R}(\varepsilon)$.

Proof: Fix $(\varepsilon_1,\varepsilon_2,\delta)$ satisfying the conditions in the lemma and let $|\varphi_1\rangle^{E_1A_1}\otimes|\varphi_2\rangle^{E_2A_2}$ be the purifications of the fixed quantum inputs $\rho_1^{A_1}\otimes\rho_2^{A_2}$, with E_1,E_2 denoting the purifying reference (or the environment) systems. We need to show that for any $(R_1,R_2)\in\mathcal{R}^{QC-fb}_{inner}(\varepsilon_1,\varepsilon_2,\delta)$ (defined in (51)), there exists an (R_1,R_2,ε) one-shot MAC simulation protocol as mentioned in Definition 10.

We note that since the overall state used to evaluate $\mathcal{R}_{inner}(\varepsilon_1, \varepsilon_2, \delta)$ is a CQ state τ given in (56) satisfying the simulation criterion, we have:

$$\eta^{E_1E_2X_1X_2Y} = \mathrm{Tr}_{U_1,U_2}\, \tau^{E_1E_2X_1X_2U_1U_2Y}$$
$$\Rightarrow p_{X_1}(x_1)p_{X_2}(x_2)q_{Y|\vec{X}}(y|\vec{x}) = \sum_{\vec{u}} p_{X_1}(x_1)p_{X_2}(x_2)p_{U_1|X_1}(u_1|x_1)p_{U_2|X_2}(u_2|x_2)p_{Y|\vec{U}}(y|\vec{u})\ . \tag{56}$$

We henceforth consider the joint distribution $p_{X_1,X_2,U_1,U_2,Y}$ in (56) throughout the proof. We will show that (R_1, R_2) is achievable by constructing a protocol that uses the convex split lemma from Fact 3, for each sender.

- **Encoding:** The input to the encoders $\mathcal{E}_1$ and $\mathcal{E}_2$ are the post-measurement states ${\varphi_1}^{E_1X_1X_1'}$ and $\varphi_2^{E_2X_2X_2'}$, respectively. The encoders then generate the classical auxiliary random variables U_j by post processing X_j' with the dephasing map $\mathcal{I}^{E_j} \otimes\ \mathcal{C}_j^{X_j'\to U_j}$. This map is essentially a measurement channel that measures the state on X_j' in an orthonormal basis $\{|u_j\rangle\}^{U_j}$ and outputs the classical variable U_j distributed according to the conditional distribution $p_{U_j|X_j}$, formally defined as follows:

$$\mathcal{C}_j^{X_j'\to U_j} : \left(\sum_{x_j} p_{X_j'}(x_j)\, |x_j\rangle\!\langle x_j|^{X_j'}\right) \mapsto \left(\sum_{u_j,x_j} p_{X_j'}(x_j)p_{U_j|X_j}(u_j|x_j)\, |u_j\rangle\!\langle u_j|^{U_j}\right)$$

Thus, for $j \in \{1,2\}$, the overall states $\varphi_1^{E_1X_1U_1}$ and $\varphi_2^{E_2X_2U_2}$ are given by:

$$\varphi_j^{E_jX_jU_j} := \sum_{u_j,x_j} p_{X_j}(x_j)p_{U_j|X_j}(u_j|x_j)\, |x_j\rangle\!\langle x_j|^{X_j} \otimes |u_j\rangle\!\langle u_j|^{U_j} \otimes \varphi_{x_j}^{E_j}.$$
$$\Rightarrow \varphi_j^{E_jU_j} = \sum_{u_j} p_{U_j}(u_j)\, |u_j\rangle\!\langle u_j|^{U_j} \otimes \sum_{x_j} p_{X_j|U_j}(x_j|u_j)\varphi_{x_j}^{E_j} = \sum_{u_j} p_{U_j}(u_j)\, |u_j\rangle\!\langle u_j|^{U_j} \otimes \varphi_{u_j}^{\prime E_j}, \tag{57}$$

where $\varphi_{u_j}^{\prime E_j} := \sum_{x_j} p_{X_j|U_j}(x_j|u_j)\varphi_{x_j}^{E_j}$. Note that these auxiliary random variables $U_j \sim p_{U_j|X_j}$ will satisfy the condition of (56). The resultant state to be further encoded or compressed to achieve lower rates is the purification of ${\varphi_1}^{E_1X_1U_1} \otimes {\varphi_2}^{E_2X_2U_2}$, which is given by:

$$|\varphi_j\rangle^{E_jE_j'X_jX_j''U_j\tilde{U}_j} := \sum_{x_j,u_j} \sqrt{p_{X_j}(x_j)p_{U_j|X_j}(u_j|x_j)}\, |\varphi_{x_j}\rangle^{E_jE_j'}\, |x_jx_j\rangle^{X_jX_j''}\, |u_ju_j\rangle^{U_j\tilde{U}_j}\ . \tag{58}$$

Sender j holds the registers E_j', X_j, X_j'', U_j and $\tilde{U}_j$. Now we use the *encoders* $\mathcal{E}_{j,\text{meas. comp.}} : \mathcal{S}_j \times \tilde{\mathcal{U}}_j \to [1 : 2^{R_j}]$ of the measurement compression protocol with feedback (see Definition 12 and the proof in Appendix E-B). These measurement compression encoders compress $\tilde{U}_j$ using one half of the available shared randomness to a message M_j described by R_j bits. We denote the overall encoder $\mathcal{E}_j = \mathcal{E}_{j,\text{meas. comp.}} \circ \mathcal{C}^{X_j'\to U_j}$ (see (Appendix E-B for details of $\mathcal{E}_{j,\text{meas. comp.}}$).

- **Decoding:** The decoding is composed of the following two steps:

1. The receiver first recovers U_j from the received message index and the available shared randomness. This is accomplished by using the decoders $\mathcal{D}_j : [1 : 2^{R_j}] \otimes \mathcal{S}_j \to \bar{\mathcal{U}}_j$ of the measurement compression theorem ($\mathcal{D}_{\text{meas. comp.}}$) from Definition 12 with exact details in Appendix E-B. Essentially these are the isometries that are guaranteed by Uhlmann's theorem (Fact 11) in the convex split lemma. The recovered pairs are denoted by $\bar{U}_1, \bar{U}_2$ and in effect the correlations with E_1, E_2 are "preserved".
Let the overall state after the application of $(\mathcal{D}_j \circ \mathcal{E}_j)$ be denoted as:

$$|\tilde{\varphi}_j\rangle^{E_jE_j'X_jX_j'U_j\bar{U}_j} := \sum_{x_j,u_j} \sqrt{\tilde{p}_{X_j,\bar{U}_j}(x_j,u_j)}\, |\varphi_{x_j}\rangle^{E_jE_j'}\, |x_jx_j\rangle^{X_jX_j'}\, |u_ju_j\rangle^{U_j\bar{U}_j} \tag{59}$$
$$\Rightarrow \tilde{\varphi}_j^{E_j\bar{U}_j} = \sum_{x_j,u_j} \tilde{p}_{X_j,\bar{U}_j}(x_j,u_j)\varphi_{x_j}^{E_j} \otimes |u_j\rangle\!\langle u_j|^{\bar{U}_j} \text{ and} \tag{60}$$
$$\varepsilon_j \overset{(a)}{\geq} \|\tilde{\varphi}_j - \varphi_j\|_{tvd} \geq \left\|\tilde{p}_{X_j,\bar{U}_j} - p_{X_j,U_j}\right\|_{tvd} \tag{61}$$

where (a) follows from Fact 3 for

$$R_j \geq I_{\max}^{\varepsilon_j-\delta}(E_j;U_j)_\tau + 2\log\frac{1}{\delta};\ j \in \{1,2\}. \tag{62}$$

2. The decoder $\mathcal{D}$ use these recovered classical states $\bar{U}_j$ and finally outputs $Y \sim p_{Y|\bar{U}_1,\bar{U}_2}$.

- **Analysis of the code:** We now show that the code defined above using $\mathcal{E}_1 \otimes \mathcal{E}_2$ as the encoder and $p_{Y|U_1,U_2} \circ (\mathcal{D}_1 \otimes \mathcal{D}_2)$ as the decoder satisfies the simulation constraint and hence is a valid simulation code for CS-QC MAC with feedback. And we finally evaluate the rate of this code. For this, we first recall that the actual channel output $\eta^{YE_1E_2X_1X_2}$ from (46) can be written as:

$$\eta^{YE_1E_2X_1X_2} := \sum_{\vec{x},y} q_{Y|\vec{X}}(y|\vec{x})p_{X_1}(x_1)p_{X_2}(x_2)\,|y\rangle\langle y|^Y \otimes \varphi_{x_1}^{E_1} \otimes \varphi_{x_2}^{E_2} \otimes |x_1\rangle\langle x_1|^{X_1} \otimes |x_2\rangle\langle x_2|^{X_2} \tag{63}$$

with an extension:

$$\tau^{YE_1E_2X_1X_2U_1U_2} := \sum_{\vec{x},\vec{u},y} p_{Y|\vec{U}}(y|\vec{u})p_{X_1}(x_1)p_{U_1|X_1}(u_1|x_1)p_{X_2}(x_2)p_{U_2|X_2}(u_2|x_2)\,|y\rangle\langle y|^Y \otimes \varphi_{x_1}^{E_1} \otimes \varphi_{x_2}^{E_2} \otimes$$
$$|x_1\rangle\langle x_1|^{X_1} \otimes |x_2\rangle\langle x_2|^{X_2} \otimes |u_1\rangle\langle u_1|^{U_1} \otimes |u_2\rangle\langle u_2|^{U_2} \text{ such that}$$
$$\tau^{E_1E_2X_1X_2Y} = \eta^{E_1E_2X_1X_2Y} \Rightarrow p_{X_1,X_2,Y} = p_{X_1}p_{X_2}q_{Y|X_1,X_2}\ . \tag{64}$$

where (64) holds due to the block diagonal structure of the CQ states η and τ. Validity of the simulation constraint: Let the overall final state of the protocol be:

$$\tilde{\tau}^{YE_1E_2X_1X_2U_1U_2} := \sum_{\vec{x},\vec{u},y} p_{Y|\vec{U}}(y|\vec{u})\tilde{p}_{X_1,U_1}(x_1,u_1)\tilde{p}_{X_2,U_2}(x_2,u_2)\,|y\rangle\langle y|^Y \otimes \varphi_{x_1}^{E_1} \otimes \varphi_{x_2}^{E_2} \otimes$$
$$|x_1\rangle\langle x_1|^{X_1} \otimes |x_2\rangle\langle x_2|^{X_2} \otimes |u_1\rangle\langle u_1|^{U_1} \otimes |u_2\rangle\langle u_2|^{U_2}\,. \tag{65}$$

We can now apply triangle inequality to obtain the following bound:

$$\begin{aligned}\left\|\tilde{\tau}^{\,E_1E_2X_1X_2Y} - \eta^{E_1E_2X_1X_2Y}\right\|_{tvd} &\leq \left\|\tilde{\tau}^{E_1E_2X_1X_2Y} - \tau^{E_1E_2X_1X_2Y}\right\|_{tvd} + \left\|\tau^{E_1E_2X_1X_2Y} - \eta^{E_1E_2X_1X_2Y}\right\|_{tvd}\\ &\overset{(i)}{\leq} \left\|\tilde{\tau}^{E_1E_2X_1X_2Y} - \tau^{E_1E_2X_1X_2Y}\right\|_{tvd}\\ &\overset{(ii)}{\leq} \varepsilon_1 + \varepsilon_2\ ,\end{aligned} \tag{66}$$

where (i) follows from (64) and (ii) holds due to the following analysis:

$$\begin{aligned}\|\tilde{\tau} - \tau\|_{tvd} &= \Bigg\| \sum_{y,\vec{u},\vec{x}} p_{Y|\vec{U}}(y|\vec{u})\Big[\tilde{p}_{X_1,U_1}(x_1,u_1)\tilde{p}_{X_2,U_2}(x_2,u_2) - p_{X_1,U_1}(x_1,u_1)p_{X_2,U_2}(x_2,u_2)\Big]\,|y\rangle\langle y| \otimes\\ &\qquad |u_1\rangle\langle u_1| \otimes |u_2\rangle\langle u_2| \otimes |x_1\rangle\langle x_1| \otimes |x_2\rangle\langle x_2| \otimes \varphi_{x_1}^{E_1} \otimes \varphi_{x_2}^{X_2}\Bigg\|_{tvd}\\ &= \frac{1}{2}\sum_{y,\vec{u},\vec{x}} p_{Y|\vec{U}}(y|\vec{u})\left|\Big[\tilde{p}_{X_1,U_1}(x_1,u_1)\tilde{p}_{X_2,U_2}(x_2,u_2) - p_{X_1,U_1}(x_1,u_1)p_{X_2,U_2}(x_2,u_2)\Big]\right|\\ &= \frac{1}{2}\sum_{\vec{u},\vec{x}} p_{\vec{X}}(\vec{x})\Big[\tilde{p}_{X_1,U_1}(x_1,u_1)\tilde{p}_{X_2,U_2}(x_2,u_2) - p_{X_1,U_1}(x_1,u_1)p_{X_2,U_2}(x_2,u_2)\Big]\\ &\leq \frac{1}{2}\sum_{\vec{u},\vec{x}}[|\tilde{p}_{X_1,U_1}(x_1,u_1)\tilde{p}_{X_2,U_2}(x_2,u_2) - p_{X_1,U_1}(x_1,u_1)\tilde{p}_{X_2,U_2}(x_2,u_2)| +\\ &\qquad |p_{X_1,U_1}(x_1,u_1)\tilde{p}_{X_2,U_2}(x_2,u_2) - p_{X_1,U_1}(x_1,u_1)p_{X_2,U_2}(x_2,u_2)|]\\ &\overset{(a)}{\leq} \frac{1}{2}\sum_{\vec{u},\vec{x}}[|\tilde{p}_{X_1,U_1}(x_1,u_1) - p_{X_1,U_1}(x_1,u_1)||\tilde{p}_{X_2,U_2}(x_2,u_2)| +\\ &\qquad |p_{X_1,U_1}(x_1,u_1)||\tilde{p}_{X_2,U_2}(x_2,u_2) - p_{X_2,U_2}(x_2,u_2)|]\\ &\overset{(b)}{\leq} \varepsilon_1 + \varepsilon_2\ ,\end{aligned}$$

where (a) holds by triangle inequality and (b) holds by using (61).
We have thus shown that the protocol with $\mathcal{E}_1 \otimes \mathcal{E}_2 = (\mathcal{E}_{1,\text{meas. comp.}} \circ \mathcal{C}^{X_1 \to U_1}) \otimes (\mathcal{E}_{2,\text{meas. comp.}} \circ \mathcal{C}^{X_2 \to U_2})$ as encoders and $\mathcal{D}^{X_1 X_2 \to Y} := p_{Y|U_1,U_2} \circ (\mathcal{D}_{1,\text{meas. comp.}} \otimes \mathcal{D}_{2,\text{meas. comp.}})$ as the decoder, is an $(R_1, R_2, \varepsilon_1 + \varepsilon_2)$-simulation code for CS-QC MAC with feedback provided (62) holds.

■

C. Converse

Lemma 3.2: Let $\varepsilon_1, \varepsilon_2 \in (0, 0.5)$ and $\varepsilon = \varepsilon_1 + \varepsilon_2$. Then, $\mathcal{R}(\varepsilon) \subseteq \mathcal{R}^{QC-fb}_{outer}(\varepsilon_1, \varepsilon_2)$.

Proof: Let $(\varepsilon_1, \varepsilon_2)$ and ε satisfy the conditions of the lemma. We need to show that any (R_1, R_2, ε) CS-QC MAC simulation protocol according to Definition 10 has $(R_1, R_2) \in \mathcal{R}^{QC-fb}_{outer}(\varepsilon_1, \varepsilon_2)$ (defined in (53)). Let the output state of the channel with feedback be $\eta^{E_1 E_2 X_1 X_2 Y}$ and let $(R_1, R_2, \varepsilon_1 + \varepsilon_2)$ simulation code produce the state $\tau'^{\,E_1 E_2 X_1 X_2 Y}$ having the form of (49) and satisfying the following simulation constraint:

$$\left\| \tau'^{\,E_1 E_2 X_1 X_2 Y} - \eta^{E_1 E_2 X_1 X_2 Y} \right\|_{tvd} \leq \varepsilon = \varepsilon_1 + \varepsilon_2.$$

Let the respective purifications of the post-measurement state after measurement $(\Lambda \otimes \Gamma)$ be given by $|\varphi\rangle^{E_1 E_1' X_1 X_1'} \otimes |\psi\rangle^{E_2 E_2' X_2 X_2'}$ and the shared randomness be denoted by $S_1^{A_1' \mathring{A}_1'} \otimes S_2^{A_2' \mathring{A}_2'}$. The Stinespring isometry of the encoders $V_{\mathcal{E}_1}^{X_1' A_1' \to M_1 M_1' \bar{X}_1 \bar{A}_1} \otimes V_{\mathcal{E}_2}^{X_2' A_2' \to M_2 M_2' \bar{X}_2 \bar{A}_2}$ creates the pure states (where $\bar{A}_j \cong A_j',\ \bar{X}_j \cong X_j$):

$$\begin{aligned}
|\nu_1'\rangle^{E_1 E_1' M_1 M_1' \bar{A}_1 \mathring{A}_1' A_1'' \mathring{A}_1'' X_1 \bar{X}_1} &:= \sum_{x_1, m_1, s_1} \sqrt{p_{M_1|X_1,S_1}(m_1|x_1,s_1) p_{X_1}(x_1) p_{S_1}(s_1)}\, |m_1 m_1\rangle^{M_1 M_1'} |x_1 x_1\rangle^{X_1 \bar{X}_1} \otimes \\
&\qquad |s_1 s_1\rangle^{\bar{A}_1 A_1''} |s_1 s_1\rangle^{\mathring{A}_1' \mathring{A}_1''} |\varphi_{x_1}\rangle^{E_1 E_1'} \qquad (67)\\
|\nu_2'\rangle^{E_2 E_2' M_2 M_2' \bar{A}_2 \mathring{A}_2' A_2'' \mathring{A}_2'' X_2 \bar{X}_2} &:= \sum_{x_2, m_2, s_2} \sqrt{p_{M_2|X_2,S_2}(m_2|x_2,s_2) p_{X_2}(x_2) p_{S_2}(s_2)}\, |m_2 m_2\rangle^{M_2 M_2'} |x_2 x_2\rangle^{X_2 \bar{X}_2} \otimes \\
&\qquad |s_2 s_2\rangle^{\bar{A}_2 A_2''} |s_2 s_2\rangle^{\mathring{A}_2' \mathring{A}_2''} |\varphi_{x_2}\rangle^{E_2 E_2'}. \qquad (68)
\end{aligned}$$

In summary, the encoders produce the following output states:

$$\nu_1'^{\,E_1 M_1 \mathring{A}_1'} = \sum_{x_1, m_1, s_1} p_{M_1|X_1,S_1}(m_1|x_1,s_1) p_{X_1}(x_1) p_{S_1}(s_1) |m_1\rangle\langle m_1|^{M_1} \otimes |s_1\rangle\langle s_1|^{\mathring{A}_1'} \otimes \varphi_{x_1}^{E_1}, \qquad (69)$$

$$\nu_2'^{\,E_2 M_1 \mathring{A}_2'} = \sum_{x_2, m_2, s_2} p_{M_2|X_2,S_2}(m_2|x_2,s_2) p_{X_2}(x_2) p_{S_2}(s_2) |m_2\rangle\langle m_2|^{M_2} \otimes |s_2\rangle\langle s_2|^{\mathring{A}_2'} \otimes \varphi_{x_2}^{E_2}. \qquad (70)$$

Now, similar to the converse part of Theorem 1 in Lemma 1.2 we will identify the auxiliary random variables U_j from the classical message M_j and parts of shared randomness. In order to do so, we will curtail the states ν_j' in their eigen basis, so that the resultant state has a similar CQ form and is appropriately close to ν'. We now give the formal description of this intuition.
For every $\vec{x}$, we now define the 'bad' set $\mathcal{C}_{\vec{x}}$ as the complement of the following set:

$$\bar{\mathcal{C}}_{\vec{x}} := \left\{ (\vec{m}, \vec{s}) : p'_{M_j|S_j,X_j}(m_j|s_j,x_j) \geq \frac{\varepsilon_j}{|\mathcal{M}_j|}, j = 1,2 \right\}. \qquad (71)$$

Note that this is very similar to the set defined in (14) for the classical case. Henceforth, the analysis is almost fully classical from this step, except that the rates will be evaluated with respect to the CQ state τ to be identified in (75).
Consider the following joint distribution on $\vec{X}, \vec{M}, \vec{S}, Y$:

$$p'_{\vec{X},\vec{S},\vec{M},Y}(\vec{x}, \vec{s}, \vec{m}, y) := \begin{cases} \bigotimes_{j=1}^{2} \frac{p_{X_j}(x_j) p_{M_j,S_j|X_j}(m_j,s_j|x_j)}{\mathbb{P}_p(\bar{\mathcal{C}}_{x_j})} P_{Y|\vec{M},\vec{S}}(y|\vec{m}\vec{s}) & (\vec{m}, \vec{s}) \in \bar{\mathcal{C}}_{\vec{x}} \\ 0 & otherwise\ . \end{cases} \qquad (72)$$

Now using the definitions from equations (71), (72) we identify the classical auxiliary random variable for every x_1 as $U_1 = (M_1, S_1)\mathbb{1}_{\bar{\mathcal{C}}_{x_1}} \sim \frac{p_{U_1|X_1}(u_1|x_1)\mathbb{1}_{\bar{\mathcal{C}}_{x_1}}}{\mathbb{P}_p(\bar{\mathcal{C}}_{x_1})}$ and $U_2 = (M_2, S_2)\mathbb{1}_{\bar{\mathcal{C}}_{x_2}} \sim \frac{p_{U_2|X_2}(u_2|x_2)\mathbb{1}_{\bar{\mathcal{C}}_{x_2}}}{\mathbb{P}_p(\bar{\mathcal{C}}_{x_2})}$. We thus define the encoded states ν_1, ν_2 with U_1, U_2 as:

$$\begin{aligned}
\nu_1^{E_1U_1A_1''X_1} &:= \sum_{\substack{x_1\in\mathcal{X}_1,\\ (m_1,s_1)\in\bar{\mathcal{C}}_{x_1}}} p_{X_1}(x_1)p_{S_1}(s_1)p'_{M_1|S_1,X_1}(m_1|s_1,x_1)\,|x_1\rangle\langle x_1|^{X_1}\otimes|m_1,s_1\rangle\langle m_1,s_1|^{U_1}\otimes|s_1\rangle\langle s_1|^{A_1''}\otimes\varphi_{x_1}^{E_1}\\
&= \sum_{x_1,u_1} p_{X_1}(x_1)p'_{U_1|X_1}(u_1|x_1)\,|x_1\rangle\langle x_1|^{X_1}\otimes|u_1\rangle\langle u_1|^{U_1}\otimes|s_1\rangle\langle s_1|^{A_1''}\otimes\varphi_{x_1}^{E_1}\\
\nu_2^{E_2U_2A_2''X_2} &:= \sum_{\substack{x_2\in\mathcal{X}_2,\\ (m_2,s_2)\in\bar{\mathcal{C}}_{x_2}}} p_{X_2}(x_2)p_{S_2}(s_2)p_{M_2|S_2,X_2}(m_2|s_2,x_2)\,|x_2\rangle\langle x_2|^{X_2}\otimes|m_2,s_2\rangle\langle m_2,s_2|^{U_2}\otimes|s_2\rangle\langle s_2|^{B_2''}\otimes\psi_{x_2}^{E_2}\\
&= \sum_{x_2,u_2} p_{X_2}(x_2)p'_{U_2|X_2}(u_2|x_2)\,|x_2\rangle\langle x_2|^{X_2}\otimes|u_2\rangle\langle u_2|^{U_2}\otimes|s_2\rangle\langle s_2|^{A_2''}\otimes\varphi_{x_2}^{E_2}
\end{aligned}$$

The probability of the set $\mathcal{C}_{\vec{x}}$ is upper bounded by $\varepsilon_1+\varepsilon_2$ similar to (15). Hence, $\mathbb{P}(\bar{\mathcal{C}}_{\vec{x}}) \geq 1-(\varepsilon_1+\varepsilon_2)$. We now show that $\left\|\nu_j - \nu_j'\right\|_{tvd} \leq \varepsilon_j$ for $j\in\{1,2\}$. For this, first note that for every $\vec{x}$ the set $\bar{\mathcal{C}}_{\vec{x}}$ can be seen as Cartesian product of the sets $\bar{\mathcal{C}}_{x_j} := \left\{(m_j,s_j) : p_{M_j,S_j}(m_j,s_j) > \frac{\varepsilon_j}{|\mathcal{M}_j|}\right\}$. Also, $\mathbb{P}_p(\bar{\mathcal{C}}_{x_j}) \geq 1-\varepsilon_j$. Thus:

$$\begin{aligned}
\left\|\nu_j' - \nu_j\right\|_1 &= \sum_{x_j}\sum_{m_j,s_j\in\bar{\mathcal{C}}_{x_j}}\left|p_{X_j}(x_j)\left[\frac{p_{M_j,S_j|X_j}(m_j,s_j|x_j)}{\mathbb{P}_p(\bar{\mathcal{C}}_{x_j})} - p_{M_j,S_j|X_j}(m_j,s_j|x_j)\right]\right| +\\
&\quad \sum_{x_j}\sum_{m_j,s_j\in\mathcal{C}_{x_j}} p_{X_j}(x_j)\left[p_{M_j,S_j|X_j}(m_j,s_j|x_j)\right]\\
&= \sum_{x_j} p_{X_j}(x_j)\left[\mathbb{P}_p(\bar{\mathcal{C}}_{x_j})\left(\frac{1}{\mathbb{P}_p(\bar{\mathcal{C}}_{x_j})}-1\right) + \mathbb{P}_p(\mathcal{C}_{x_j})\right]\\
&= 2\sum_{x_j} p_{X_j}(x_j)\mathbb{P}_p(\mathcal{C}_{x_j}) \leq 2\varepsilon_j,\\
&\Rightarrow \left\|\nu_j' - \nu_j\right\|_{tvd} \leq \varepsilon_j\ .
\end{aligned} \tag{73}$$

Similar to (67), we define the purifications of the states ν_j' as follows:

$$\begin{aligned}
|\nu_j'\rangle^{E_jE_j'M_jM_j'\bar{A}_j\mathring{A}_j'A_j''\mathring{A}_j''X_j\bar{X}_j} &:= \sum_{x_j}\sum_{(m_j,s_j)\in\bar{\mathcal{C}}_{x_j}}\sqrt{p'_{M_j|X_j,S_j}(m_j|x_j,s_j)p_{X_j}(x_j)p_{S_j}(s_j)}\,|m_jm_j\rangle^{M_jM_j'}|x_jx_j\rangle^{X_j\bar{X}_j}\\
&\qquad\otimes|s_js_j\rangle^{\bar{A}_jA_j''}|s_js_j\rangle^{\mathring{A}_j'\mathring{A}_j''}|\varphi_{x_j}\rangle^{E_jE_j'},\\
\Rightarrow |\nu_j'\rangle^{E_jE_j'U_jU_j'\mathring{A}_j'\mathring{A}_j''X_j\bar{X}_j} &:= \sum_{x_j}\sum_{(m_j,s_j)\in\bar{\mathcal{C}}_{x_j}}\sqrt{p'_{U_j|X_j}(u_j|x_j)p_{X_j}(x_j)}\,|u_ju_j\rangle^{U_jU_j'}|x_jx_j\rangle^{X_j\bar{X}_j}|s_js_j\rangle^{\mathring{A}_j'\mathring{A}_j''}|\varphi_{x_j}\rangle^{E_jE_j'}
\end{aligned} \tag{74}$$

The overall state after the action of the decoder acting on $|\nu_1\rangle\otimes|\nu_2\rangle$ is denoted (after tracing out all but $E_1E_2U_1U_2X_1X_2Y$ subsystems) by $\tau^{E_1E_2U_1U_2X_1X_2Y}$:

$$\begin{aligned}
\tau^{E_1E_2U_1U_2X_1X_2Y} &:= \mathcal{D}^{U_1U_2\to Y}(\nu_1\otimes\nu_2)\\
&= \sum_{\vec{x},y}\sum_{\vec{m},\vec{s}\in\bar{\mathcal{C}}} p_{\vec{X}}(\vec{x})\frac{p_{\vec{M},\vec{S}|\vec{X}}(\vec{m},\vec{s}|\vec{x})}{\mathbb{P}_p(\bar{\mathcal{C}})}p_{Y|\vec{M},\vec{S}}(y|\vec{m},\vec{s})\,|y\rangle\langle y|^Y\otimes|x_1\rangle\langle x_1|^{X_1}\otimes|x_2\rangle\langle x_2|^{X_2}\otimes\\
&\qquad\otimes|m_1,s_1\rangle\langle m_1,s_1|^{U_1}\otimes|m_2,s_2\rangle\langle m_2,s_2|^{U_2}\otimes\varphi_{x_1}^{E_1}\otimes\varphi_{x_2}^{E_2}
\end{aligned}$$

We thus have that:

$$\begin{aligned}
&\left\|\tau'^{E_1E_2X_1X_2Y} - \tau^{E_1E_2X_1X_2Y}\right\|_{tvd} \qquad (75)\\
&= \left\|\mathcal{D}\left(\nu_1'^{E_1X_1M_1M_1'\mathring{A}_1'\mathring{A}_1''}\otimes\nu_2'^{E_2X_2M_2M_2'\mathring{A}_2'\mathring{A}_2''} - \nu_1^{E_1X_1M_1M_1'\mathring{A}_1'\mathring{A}_1''}\otimes\nu_2^{E_2X_2M_2M_2'\mathring{A}_2'\mathring{A}_2''}\right)\right\|_{tvd}
\end{aligned}$$

$$\leq \left\| \nu_1' \otimes \nu_2' - \nu_1 \otimes \nu_2 \right\|_{tvd}$$
$$\overset{(a)}{\leq} \sum_{j=1}^{2} \left\| \nu_j' - \nu_j \right\|_{tvd}$$
$$\overset{(b)}{\leq} \varepsilon_1 + \varepsilon_2, \tag{76}$$

where (a) holds because of triangle inequality and (b) follows from (73). Finally, using the simulation constraint $\left\| \tau'^{E_1E_2X_1X_2Y} - \eta^{E_1E_2X_1X_2Y} \right\|_{tvd} \leq \varepsilon_1 + \varepsilon_2$ and the triangle inequality again, we get that

$$\left\| \tau^{E_1E_2X_1X_2Y} - \eta^{E_1E_2X_1X_2Y} \right\|_{tvd} \leq 2(\varepsilon_1 + \varepsilon_2).$$

We now evaluate the rate of the code. First we notice that $\tau^{E_jU_j} = \tau^{E_jA_j''M_j} = \nu_j^{E_jA_j''M_j}$ and $\tau'^{E_jA_j''M_j} = \nu_j'^{E_jA_j''M_j}$ for $j \in \{1,2\}$. Hence, we have

$$\tau^{E_jU_j} = \nu_j^{E_jA_j''M_j} \overset{(a)}{\leq} |\mathcal{M}_j| \left(\nu_j^{E_j} \otimes \tilde{\nu}_j^{A_j''M_j} \right), \tag{77}$$

where (a) holds due to

$$\nu_j^{E_jM_jA_j''} = \sum_{x_j} p_{X_j}(x_j) \varphi_{x_j}^{E_j} \otimes \sum_{m_j, s_j \in \bar{\mathcal{C}}_{x_j}} p_{S_j}(s_j) p_{M_j|X_j,S_j}(m_j|x_j,s_j) \left| m_j \middle\rangle\!\middle\langle m_j \right|^{M_j} \otimes \left| s_j \middle\rangle\!\middle\langle s_j \right|^{A_j''}$$
$$\leq \sum_{x_j} p_{X_j}(x_j) \varphi_{x_j}^{E_j} \otimes \sum_{m_j, s_j} p_{S_j}(s_j) \left| m_j \middle\rangle\!\middle\langle m_j \right|^{M_j} \otimes \left| s_j \middle\rangle\!\middle\langle s_j \right|^{A_j''},$$

In the above, we used

$$p'_{M_j|S_j,X_j}(m_j|s_j,x_j) \leq 1, \ \forall (x_j, s_j, m_j),$$
$$\text{and } \tilde{\nu}_j^{A_j''M_j} := \sum_{m_j,s_j} \frac{p_{S_j}(s_j)}{|\mathcal{M}_j|} \left| m_j \middle\rangle\!\middle\langle m_j \right|^{M_j} \otimes \left| s_j \middle\rangle\!\middle\langle s_j \right|^{A_j''}.$$

Hence, from (77) and Definition 4 of the quantum smoothed max-mutual information, we have

$$R_j = \log |\mathcal{M}_j| \geq I_{\max}^{\varepsilon_j}(E_j; U_j)_\tau.$$

We finally state the cardinality bounds of $\mathcal{U}_1, \mathcal{U}_2$ as Lemma 3.3 below. This completes the proof of the converse. ■

Lemma 3.3: The cardinalities of $\{\mathcal{U}_1, \mathcal{U}_2\}$ for $\mathcal{R}_{outer}^{QC-fb}$ can be upper bounded as:

$$|\mathcal{U}_j| \leq |\mathcal{X}_1||\mathcal{X}_2||\mathcal{Y}|; \quad \text{ for } j \in \{1,2\}. \tag{78}$$

The proof is very similar to that of Lemma 1.3, as U_1, U_2 are classical, and is given in Appendix B-C.

D. Asymptotic iid expansion

We now give the asymptotic iid characterization of the rate region for simulating a CS-QC MAC with feedback.

Corollary 3.1: Consider the classical scrambling QC-MAC $\mathcal{N}^{A_1A_2 \to YX_1X_2}$ with feedback, given by (46) and inputs $\rho_1^{A_1} \otimes \rho_2^{A_2}$ with their respective purifications of form $|\varphi_1\rangle^{E_1A_1} \otimes |\varphi_2\rangle^{E_2A_2}$. The rate region for simulating $\mathcal{N}^{A_1A_2 \to YX_1X_2}$ using infinite shared randomness between each sender-receiver pair and classical communication over links of (R_1, R_2) is given by:

$$\mathcal{R}_{iid}^{QC-fb} = cl \left\{ \bigcup_{\substack{\tau^{E_1E_2X_1X_2U_1U_2Y}: \\ (E_1,E_2) \to (X_1,X_2) \to (U_1,U_2) \to Y \\ \& \ \tau^{E_1E_2X_1X_2Y} = \eta^{E_1E_2X_1X_2Y} \\ |\mathcal{U}_1|, |\mathcal{U}_2| \leq |\mathcal{X}_1||\mathcal{X}_2||\mathcal{Y}|}} \{(R_1,R_2) : R_j \geq I(E_j;U_j)_\tau, \quad j \in \{1,2\}\} \right\}. \tag{79}$$

Proof: **Asymptotic iid Inner Bound:** The one-shot inner bound can be straight away extended to obtain the optimal asymptotic iid rate region. Let $(R_1, R_2) \in \mathcal{R}_{iid}^{QC-fb}$ be such that for any $\zeta > 0$,

$$R_j \geq I(E_j; U_j)_\tau + \zeta, \tag{80}$$

for some $\tau_{E_1,E_2,X_1,X_2,U_1,U_2,Y}$ having the form described by (49) and satisfying $\tau_{E_1,E_2,Y} = \eta_{E_1,E_2,Y}$. Consider the following n-letter iid extension of τ, defined as $\tau^{(n)}_{E_1E_2X_1X_2U_1U_2Y} := \tau^{\otimes n}_{E_1E_2X_1X_2U_1U_2Y}$. Note that

$$\tau^{(n)}_{E_1E_2Y} = \tau^{\otimes n}_{E_1E_2Y} = \eta^{\otimes n}_{E_1E_2Y}. \tag{81}$$

Now, the AEP for the smoothed max-mutual information (see (101)of Fact 8) yields

$$\lim_{n\to\infty} \frac{1}{n}\left[I_{\max}^{\varepsilon_j-\delta}(E_j^n, U_j^n)_{\tau^{(n)}} + 2\log\left(\frac{1}{\delta}\right)\right] = I(E_j; U_j)_\tau,$$

which by (80) means that

$$nR_j \geq I_{\max}^{\varepsilon_j-\delta}(E_j^n, U_j^n)_{\tau^{(n)}} + 2\log\frac{1}{\delta}, \tag{82}$$

for all sufficiently large n (depending on ζ). This along with (81) implies $\mathcal{R}_{iid}^{QC-fb} \subseteq \left(\mathcal{R}_{inner}^{QC-fb}\right)^{(n)}(\varepsilon_1, \varepsilon_2)$, where

$$\left(\mathcal{R}_{inner}^{QC-fb}\right)^{(n)}(\varepsilon_1, \varepsilon_2) = \left\{(R_1, R_2) : nR_j \geq I_{\max}^{\varepsilon_j-\delta}(E_j^n; U_j^n)_{\tau^{(n)}} + 2\log\frac{1}{\delta};\ \text{for } j \in \{1, 2\}\right\}. \tag{83}$$

Asymptotic iid Outer Bound: In order to prove the converse, for any $\varepsilon \in (0, 1)$ we first define the following ε-approximate iid region as follows:

$$\mathcal{R}_{iid}^{QC-fb}(\varepsilon) := \{(R_1, R_2) : R_j \geq I(E_j; U_j)_\tau, \forall\, \tau_{E_1,E_2,X_1,X_2,U_1,U_2,Y} \text{ is of form given in (49)}$$
$$\text{such that } \left\|\tau^{E_1,E_2,Y} - \eta^{E_1,E_2,Y}\right\|_{tvd} \leq \varepsilon\}. \tag{84}$$

We will now use the extension of the one-shot converse of Lemma 3.2. In order to do so, for any $\varepsilon_1, \varepsilon_2, \varepsilon \in (0, 1)$ such that $\max\{\varepsilon_1, \varepsilon_2\} \leq \varepsilon/4$, let $\left(\mathcal{R}_{outer}^{QC-fb}\right)^{(n)}(\varepsilon_1, \varepsilon_2, \varepsilon)$ be the n-fold extension of the region $\mathcal{R}_{outer}^{QC-fb}(\varepsilon_1, \varepsilon_2, \varepsilon)$ with respect to the iid inputs $|\varphi_j\rangle^{\otimes n}_{E_jA_j}$ (for $j \in \{1, 2\}$) and general auxiliary random variables $U_j^n \sim p_{U_j^n|X_j^n}$. Suppose $(R_1, R_2) \in \left(\mathcal{R}_{outer}^{QC-fb}\right)^{(n)}(\varepsilon_1, \varepsilon_2, \varepsilon)$ with

$$\tau^{(n)}_{E_1E_2X_1X_2U_1U_2Y} := \sum_{\substack{x_1^n, x_2^n\\ u_1^n, u_2^n, y^n}} p_{X_1}^{\otimes n}(x_1) p_{U_1^n|X_1^n}(u_1^n|x_1^n)\, |x_1\rangle\langle x_1|^{\otimes n}_{X_1} \otimes |u_1^n\rangle\langle u_1^n|_{U_1^n}\, p_{X_2}^{\otimes n}(x_2) p_{U_2^n|X_2^n}(u_2^n|x_2^n)\, |x_2\rangle\langle x_2|^{\otimes n}_{X_2}$$
$$\otimes |u_2^n\rangle\langle u_2^n|_{U_2^n} \otimes (\varphi_{x_1})^{\otimes n}_{E_1} \otimes (\varphi_{x_2})^{\otimes n}_{E_2} \otimes p_{Y^n|U_1^n,U_2^n}(y^n|u_1^n, u_2^n)\, |y^n\rangle\langle y^n|_{Y^n}$$

be the state induced by any n-fold simulation code. Suppose $(R_1, R_2) \in \left(\mathcal{R}_{outer}^{QC-fb}\right)^{(n)}(\varepsilon_1, \varepsilon_2, \varepsilon)$ satisfying

$$\left\|\left(\tau^{(n)}\right)^{E_1^nE_2^nX_1^nX_2^nY^n} - \left(\eta^{E_1E_2X_1X_2Y}\right)^{\otimes n}\right\|_{tvd} \leq \varepsilon \text{ (as } 2(\varepsilon_1 + \varepsilon_2) \leq \varepsilon). \tag{85}$$

Then,

$$\begin{aligned}
nR_j &\geq I_{\max}^{\varepsilon_j}(E_j^n; U_j^n)_{\tau^{(n)}}\\
&\overset{(a)}{=} I_{\max}(E_j^n; U_j^n)_{\tau'^{(n)}}\\
&\overset{(b)}{\geq} I(E_j^n; U_j^n)_{\tau'^{(n)}}\\
&\overset{(c)}{\geq} I(E_j^n; U_j^n)_{\tau^{(n)}} - 2\varepsilon_j \log|\mathcal{E}_j|^n - 2h_2\left(\frac{\varepsilon_j}{1+\varepsilon_j}\right)\\
&\overset{(d)}{\geq} nI(E_j; U_j)_{\tau_{E_jU_j}} - 2\varepsilon_j \log|\mathcal{E}_j|^n - 2h_2\left(\frac{\varepsilon_j}{1+\varepsilon_j}\right),
\end{aligned}$$

$$\Rightarrow \lim_{\varepsilon_j\to 0}\lim_{n\to\infty}\frac{I_{\max}^{\varepsilon_j}(E_j^n;U_j^n)_{\tau^{(n)}}}{n} \geq \lim_{\varepsilon_j\to 0}\lim_{n\to\infty}\left[\frac{nI(E_j;U_j)_{\tau_{E_jU_j}} - 2\varepsilon_j\log|\mathcal{E}_j|^n - g(\varepsilon_j)}{n}\right]$$
$$= I(E_j;U_j)_{\tau_{E_jU_j}},$$

where (a) holds by choosing $\tau'^{(n)} \in \mathcal{B}^{\varepsilon_j}(\tau^{(n)}_{E_jU_j})$ to be the optimizer for $I_{\max}^{\varepsilon_j}(E_j^n;U_j^n)_{\tau^{(n)}_{E_jU_j}}$; (b) holds by the fact the $I_{\max}(E;U)_{\tau'} \geq I(E;U)_{\tau'}$ for any state $\tau'_{E,U}$; (c) follows from continuity of mutual information from Fact 6 with $g(\varepsilon_j) = 2h_2\left(\frac{\varepsilon_j}{1+\varepsilon_j}\right)$; (d) follows by Proposition 2. This implies

$$R_j \geq I(E_j;U_j)_{\tau}.$$

Note that (85) and monotonicity of trace distance implies that $\left\|\tau^{E_1E_2X_1X_2Y} - \eta^{E_1E_2X_1X_2Y}\right\|_{tvd} \leq \varepsilon$ (see also (99)). Hence, we have shown that in the asymptotic iid limit:

$$\lim_{\varepsilon_1,\varepsilon_2\to 0}\lim_{n\to\infty}\left(\mathcal{R}_{outer}^{QC-fb}\right)^{(n)}(\varepsilon_1,\varepsilon_2,\delta) \subseteq \mathcal{R}^{iid}(\varepsilon). \tag{86}$$

We have thus recovered the asymptotically optimal region of [5, Theorem 1, Theorem 3] up to δ. Since, we also have that cardinalities of the auxiliary random variables are bounded, we can directly apply [14, Lemma 6] (see Fact 7 for a detailed analysis) to recover the asymptotically optimal region of Corollary 3.1 as follows:

$$\mathcal{R}_{outer}^{QC-fb} := \lim_{\varepsilon\to 0}\lim_{n\to\infty}\left(\mathcal{R}_{outer}^{QC-fb}\right)^{(n)}(\varepsilon_1,\varepsilon_2,\varepsilon) \subseteq \mathcal{R}_{iid}^{QC-fb} = \lim_{\varepsilon\to 0}\mathcal{R}_{iid}^{QC-fb}(\varepsilon).$$

Thus we have shown that in the asymptotic iid limit:

$$\mathcal{R}_{outer}^{QC-fb} \subseteq \mathcal{R}^{iid} \subseteq \lim_{n\to\infty}\left(\mathcal{R}_{inner}^{QC-fb}\right)^{(n)}(\varepsilon_1,\varepsilon_2) \subseteq \mathcal{R}_{outer}^{QC-fb}$$
$$\Rightarrow \mathcal{R}_{inner}^{QC-fb} := \lim_{\varepsilon_1,\varepsilon_2\to 0}\lim_{n\to\infty}\left(\mathcal{R}_{inner}^{QC-fb}\right)^{(n)}(\varepsilon_1,\varepsilon_2) = \mathcal{R}_{iid}^{QC-fb} = \mathcal{R}_{outer}^{QC-fb}.$$

■

VII. Conclusion

In this work, we have provided one-shot inner and outer bounds for simulating a two-independent user classical MAC, with unlimited shared randomness for fixed product inputs and also universally for two independent arbitrary inputs. Further, we have derived the corresponding generalizations to the classical scrambling QC-MAC with feedback and fixed inputs and provided a tight asymptotic iid rate region. There are plentiful of interesting connected open problems, e.g., characterizing the rate region of CS-QC MAC without feedback, as well its universal rate region. Yet, another challenging and immediate open problem is the fixed input and universal simulation of fully quantum MACs. This would call for further in depth understanding and interpretation of the global Markov condition $(A_1,A_2) \to (U_1,U_2) \to Y$ and how that can aid in viewing the MAC as two point-point channels. Another technical issue would be to bound the cardinalities of the quantum auxiliary systems U_1, U_2 , both in one-shot and asymptotic iid settings and prove a matching single-letter asymptotic iid converse.

Potential applications of our simulation techniques are in one-shot quantum multi-user rate distortion theory [16], characterizing the communication complexity of computing a function across a network [17], remotely preparing a target quantum state between several users and analyzing the related dynamical resource theory [7], and simulating other general network topologies.

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APPENDIX
ORGANIZATION

In the following appendices we give the key technical tools used to prove our results for the one-shot classical and classical scrambling QC-MAC simulation, along with the proofs of some of the main lemmas of this work. Although these tools are not new and has a quite exhaustive literature, we state the versions that we use in our proofs. We also give proofs of some of our key lemmas in some of the sections of the following appendices.

APPENDIX A
REJECTION SAMPLING AND CONVEX SPLIT LEMMA

We use the version of rejection sampling inspired from [18] and developed in [11]. We first state the accept-reject technique to shape one distribution to some other distribution of interest.

Fact 1: [11, Lemma 1] Let Y be sampled from the distribution q_Y and p_Y denote the target distribution for sampling Y. Assume p, q satisfy $p \ll q$. Let $M \geq 1$ be an integer. Suppose $Y_1, Y_2, \ldots, Y_M \sim q_Y$ be iid random variables. Define $\lambda := \max_{y \in \mathcal{Y}} \frac{p_Y(y)}{q_Y(y)} = 2^{D_{\max}(p_Y||q_Y)}$. Then there exists an algorithm, called accept-reject or rejection sampling that either outputs a random variable $\tilde{Y} \in \{Y_1, \ldots, Y_M\}$ such that for any $\varepsilon \in (0,1)$ and a large enough M satisfying:

$$2^{-\left\{\frac{M}{2^{D_{\max}(p_Y||q_Y)}}\right\}} \leq \varepsilon \text{ it holds that}$$
$$\|p_Y - \tilde{p}_Y\|_{tvd} \leq \varepsilon,$$

or outputs an abort message at termination of the algorithm.

Remark 3.1: The choice of M ensuring that the probability that the above *accept-reject* method aborts is upper bounded by:

$$\left(1 - 2^{-D_{\max}(p_Y||q_Y)}\right)^M \leq 2^{-\left\{\frac{M}{2^{D_{\max}(p_Y||q_Y)}}\right\}} \leq \varepsilon \ .$$

Thus, the accept-reject method outputs a sample distributed according to p_Y (under no abort with probability at least $1 - \varepsilon$) using $\log M \geq D_{\max}(p_Y||q_Y) + \log\log(1/\varepsilon)$ trials from shared randomness. This can easily be converted to a point-to-point channel simulation achievability protocol as stated in the Fact 2 and implemented in [11].

Using this fact we now state the specific versions of the one-shot classical point-to-point channel simulation costs for fixed input and the universal simulation criterion as the following facts:

Fact 2: Let $\varepsilon > 0$ and $\delta \in (0, \varepsilon)$. Now let $p_{X,Y}$ denote a bipartite probability distribution and $S \sim p_S$ shared randomness between sender Alice and the receiver Bob. Alice sends a message $m(x, S) \in \mathcal{M}$ to Bob, so that Bob can generate a sample $y(m, s) \overset{\varepsilon}{\sim} p_{Y|X=x}$.

(i) An achievable rate (with an almost matching converse) for the above task is given by (by fixing the input distribution in [11, Theorem 2]):

$$R := \log|\mathcal{M}| \geq I_{\max}^{\varepsilon-\delta}(X;Y)_p + \log\log\frac{1}{\delta},$$

and the resulting distribution $\tilde{p}_{X,Y}$ satisfies $\mathbb{E}_{p_X}\|\tilde{p}_{Y|X=x} - p_{Y|X=x}\|_{tvd} \leq \varepsilon$.

(ii) An achievable rate for the above task which works independent of p_X (also called universal simulation, [11, Theorem 2]) is:

$$R := \log|\mathcal{M}| \geq I_{\max}^{\varepsilon-\delta}(p_{Y|X}) + \log\log\frac{1}{\delta},$$

and the resulting distribution $\tilde{p}_{X,Y}$ satisfies $\max_x \|\tilde{p}_{Y|X=x} - p_{Y|X=x}\|_{tvd} \leq \varepsilon$. Further an almost matching converse for the universal task is given by [11, Theorem 4]:

$$R := \log|\mathcal{M}| \geq I_{\max}^{\varepsilon}(p_{Y|X}) \ .$$

Finally, for deriving our results for the QC-MAC in Section VI, we need a quantum analogue of rejection sampling called coherent rejection sampling or the convex split lemma (first formulated in [19]). Again, this is the core idea used to prove the one-shot measurement compression theorem in [20], which can be modified to carry out the CS-QC MAC simulation task, as we do here. We remark that the additive fudge term in the convex split lemma is $2\log 1/\delta$ in contrast with $\log\log 1/\delta$ in the classical setting. This is because the classical rejection sampling 'fine tunes' the input to be correlated with only the accepted sample from the shared randomness whereas the convex split step correlates input with all the registers of the shared randomness. We state the convex split lemma used in our proofs as the following fact:

Fact 3: ([20, Corollary 2] and [10, Lemma 12]) Let $\varepsilon > 0$ and $\delta \in (0, \varepsilon)$. Consider the following states:

$$\tau^{EU} := \sum_u p(u)\tau_u^E \otimes |u\rangle\langle u|^U \text{ and } \sigma^U := \sum_u q(u) |u\rangle\langle u|^U$$

such that the $supp(\tau^U) \subseteq supp(\sigma^U)$ and $\{p(u)\}_u, \{q(u)\}_u$ are probability distributions. Further, suppose q is the distribution achieving the infimum in the definition of $I_{\max}^{\varepsilon-\delta}(E;U)_\tau$ (in equation (3)). Let

$$\bar{\sigma}^{U_1U_2\dots U_n} := \sum_{u_1,u_2,\dots,u_n} \bar{q}(u_1, u_2, \dots, u_n) |u_1u_2\dots u_n\rangle\langle u_1u_2\dots u_n|^{U_1U_2\dots U_n}$$

be a quantum state satisfying $\bar{\sigma}^{U_i} = \sigma^U$ for all $i \in [1:n]$ and $\bar{q}$ be a pairwise independent probability distribution of U_1^n with each $U_i \sim q$. For the following states

$$\tau_i^{EU_1\dots U_n} := \sum_u p(u)\tau_u^E \otimes |u\rangle\langle u|^{U_i} \otimes \left(\bigotimes_{j\neq i} \bar{\sigma}^{U_j}\right),$$

$$\tau^{EU_1\dots U_n} := \sum_{i=1}^n \frac{1}{n}\tau_i^{EU_1\dots U_n},$$

and the value of parameter n satisfying

$$\log n \geq I_{\max}^{\varepsilon-\delta}(E;U)_\tau + 2\log\frac{1}{\delta},$$

it holds that

$$\left\| \tau^{EU_1\dots U_n} - \tau^E \bigotimes_{i=1}^n \sigma^{U_i} \right\|_{tvd} \leq \varepsilon. \tag{87}$$

The above fact in [20, Corollary 2] proves an upper bound of $2\varepsilon + \delta$ in (87) and $\log n$ is lower bounded by $I_{\max}^{\varepsilon}(E;U)_\tau$. The 2ε term in the distance is due to a different definition of $I_{\max}^{\varepsilon}$. More precisely, the marginal ${\tau'}^E$ of the optimal state $\tau'^{EU} \in \mathcal{B}^\varepsilon(\tau)$ for evaluating $I_{\max}^{\varepsilon}(E;U)_\tau$ need not be the same as τ^E, which is unlike our Definition 4. Thus, by using our Definition 4 we first reduce the aforementioned distance to $\varepsilon + \delta$. Further we get rid of the additional δ in the distance by choosing the radius of the ball used for smoothing to be $\varepsilon - \delta$ instead of ε (in [20, Corollary 2]), which gives us that $\log n \geq I_{\max}^{\varepsilon-\delta}(E;U)_\tau$. The same modifications were made in [10, Lemma 12] when the state τ^{EU} is fully-quantum, resulting in the similar rate expression. Even with these minor differences, the asymptotic iid limit of smoothed-$I_{\max}$ (with any of the two definitions) is $I(E;U)_\tau$ (see e.g. [12], [21]). We also note that a fully quantum version of the convex split lemma with slightly different definition of $I_{\max}^{\varepsilon}$ was also given in [22]. However this CQ version of the convex split lemma suffices for our purpose.

Appendix B
Cardinality bounds and single-letterization

A. *Cardinality of $\mathcal{U}_1, \mathcal{U}_2$ for asymptotic iid simulation*

The cardinality bounds of $\mathcal{U}_1, \mathcal{U}_2$ were proven in [5, Lemma 5] using the so-called perturbation method of [23, Claim 1] and support lemma of [24, Appendix C]. We state this result as the following fact:

Fact 4: [5, Lemma 5 and Theorem 3] Given the MAC simulation task in Definition 5 with fixed inputs $(X_1, X_2) \sim q_{X_1} \times q_{X_2}$, suppose the cost region for simulating the MAC $q_{Y|X_1,X_2}$ is given as:

$$\mathcal{R}^{iid} = \left\{(R_1, R_2) : R_j \geq I(X_j, U_j'|T)_{p'}, \text{ for } j \in \{1,2\}\right\}$$

for some overall distribution

$$p'_{X_1,X_2,U_1',U_2',Y,T}(x_1, x_2, u_1', u_2', y, t) = \pi_T(t)q_{X_1}(x_1)q_{X_2}(x_2)p_{U_1'|X_1,T}(u_1'|x_1,t)p_{U_2'|X_2,T}(u_2'|x_2,t)p_{Y|U_1',U_2',T}(y|u_1', u_2', t)$$

such that $\mathop{\mathbb{E}}_{T\sim[1:n]}\left[p_{X_1,X_2,Y|T}(x_1,x_2,y)\right] = q_{X_1}(x_1)q_{X_2}(x_2)q_{Y|X_1,X_2}(x_1,x_2)$. Then the cardinalities of random variables $\mathcal{U}'_1, \mathcal{U}'_2, \mathcal{T}$ can be restricted as follows:

$$\begin{aligned} |\mathcal{U}'_j| &\leq |\mathcal{X}_1||\mathcal{X}_2||\mathcal{Y}|, \text{ for } j \in \{1,2\}, \\ |\mathcal{T}| &\leq 2 \, . \end{aligned} \tag{88}$$

Moreover, one can show that there exists an overall distribution $p_{X_1,X_2,U_1,U_2,Y}$, with cardinalities of $\mathcal{U}_j$ as bounded above in (88) and $p_{X_j} = q_{X_j}$ (for $j \in \{1,2\}$) such that $p_{X_1,X_2,Y} = q_{X_1,X_2,Y}$ and the cost region can then be simplified as:

$$\mathcal{R}^{iid} = \{(R_1,R_2) : R_j \geq I(X_j;U_j)_p, \text{ for } j \in \{1,2\}\} \, .$$

B. Cardinality of $\mathcal{U}_1, \mathcal{U}_2$ for one-shot simulation

We first state the following fact on a characterization of the max-mutual information of two correlated random variables.

Fact 5: [9, Lemma B.5] The max-mutual information between a pair of jointly distributed random variables $(X,U) \sim p_{X,U}$ is given by:

$$I_{\max}(X;U)_p = \max_{x,u} \log\left(\frac{p_{X,U}(x,u)}{p_X(x)q_U(u)}\right) = \max_{x,u} \log\left(\frac{p_{X,U}(x,u)|\mathcal{X}|}{p_X(x)\left\{\sum_{x'} p_{U|X}(u|x')\right\}}\right) ,$$

where $q_U(u) := \sum_x \frac{p_{U|X}(u|x)}{|\mathcal{X}|}$ is the optimal distribution in the Definition 2 of $I_{\max}$.

We now prove Lemma 1.3 that upper bounds the cardinalities of auxiliary alphabets $\mathcal{U}_1, \mathcal{U}_2$.

Proof: The proof is very similar to the proof [5, Lemma 5]. We show that the cardinalities of the auxiliary alphabets can be brought down to $|\mathcal{U}_j| \leq |\mathcal{X}_1||\mathcal{X}_2||\mathcal{Y}|$ for $j \in \{1,2\}$. The essential property of any entropic quantity under consideration, that is required to bound $|\mathcal{U}_j|$, is that it should be invariant under the perturbed distribution. The entropic quantity considered here is $I_{\max}$. We note that the cost region is expressed in terms of smoothed $I_{\max}(X_j,U_j)$ and not in terms of $I(X_j;U_j)$, hence a separate proof than that of Fact 4 is needed. For this we use the *perturbation method* first developed in [8, Lemma 1 and 2] and then simplified in [23, Claim 1].
Along the lines of the perturbation method ([23, Claim 1]), we consider an optimization of the weighted sum $\nu_1 I_{\max}(X_1;U_1) + \nu_2 I_{\max}(X_2;U_2)$ for non-negative real numbers ν_1,ν_2 and come up with new auxiliary random variables with reduced alphabet sizes and not increasing the weighted sum, along with preserving the constraints on $p_{X_1,X_2,U_1,U_2,Y}$ from Lemma 1.2.

Thus, for a given $p(x_1,x_2,u_1,u_2,y)$, consider the Lyapunov perturbation $L(u_1)$ and the perturbed distribution p_ε defined by:

$$p_\varepsilon(x_1,x_2,u_1,u_2,y) := p(x_1,x_2,u_1,u_2,y)(1+\varepsilon L(u_1)) \, . \tag{89}$$

Note that the ε above can be negative. Clearly, for $p_\varepsilon(x_1,x_2,u_1,u_2,y)$ to be a valid probability distribution, it should hold that $(1+\varepsilon L(u_1)) \geq 0$ for all u_1, and $\Sigma_{u_1} p(u_1)L(u_1) = 0$. We will further consider perturbations $L(u_1)$ satisfying

$$\mathop{\mathbb{E}}_{U_1\sim p(u_1)}[L(U_1)|X_1=x_1,X_2=x_2,Y=y] = \sum_{u_1} L(u_1)p(u_1|x_1,x_2,y) = 0, \ \forall x_1,x_2,y \, . \tag{90}$$

Note that the marginal distribution of X_1,X_2 is unchanged under the above perturbation, that is, $p_\varepsilon(x_j) = p(x_j)$ for $j \in \{1,2\}$ and that $p_\varepsilon(u_1|x_1) = p(u_1|x_1)(1+\varepsilon L(u_1))$.
(89) can also be seen as a linear equation $L^T P(u|x_1,x_2,y) = 0$, where $L = \{L(u_1)\}_{u_1}$ is a vector and $[P(u_1|x_1,x_2,y)]_{|\mathcal{U}_1|\times|\mathcal{X}_1||\mathcal{X}_2||\mathcal{Y}|}$ is a stochastic matrix. Thus, by the rank-nullity theorem the range space of $[P(u_1|x_1,x_2,y)]^T_{|\mathcal{U}_1|\times|\mathcal{X}_1||\mathcal{X}_2||\mathcal{Y}|}$ is of dimension at most $|\mathcal{X}_1||\mathcal{X}_2||\mathcal{Y}|$ and hence a non- trivial (non-zero) perturbation exists as long as $|\mathcal{U}_1| > |\mathcal{X}_1||\mathcal{X}_2||\mathcal{Y}|$. Further, for sufficiently small values of $|\varepsilon|$, we also have $(1+\varepsilon L(u_1)) \geq 0$ for all u_1. A simple check ensures that this perturbation preserves the distribution $p(x_1,x_2,y)$. Similarly, a straightforward

marginalization of (89) also ensures that the global Markov constraint $(X_1, X_2) \to (U_1, U_2) \to Y$ holds also for the perturbed distribution p_ε.

Now if the distribution $p(x_1, x_2, u_1, u_2, y)$ minimize $\nu_1 I_{\max}(X_1; U_1) + \nu_2 I_{\max}(X_2; U_2)$, then for any valid perturbation, the following first derivative condition must hold:

$$\frac{d}{d\varepsilon}\left(\nu_1 I_{\max}(X_1; U_1)_{p_\varepsilon} + \nu_2 I_{\max}(X_2; U_2)_{p_\varepsilon}\right)\bigg|_{\varepsilon=0} = 0 \tag{91}$$

We now evaluate weighted sum term under the perturbed distribution $p_\varepsilon(\cdot)$. Using (89) and Fact 5 we get that $I_{\max}(X_1; U_1)_{p_\varepsilon}$ remains unaltered due to perturbation as:

$$\begin{aligned}
I_{\max}(X_1; U_1)_{p_\varepsilon} &= \max_{x_1, u_1} \log\left(\frac{p_{X_1,U_1}(x_1, u_1)(1 + \varepsilon L(u_1))|\mathcal{X}_1|}{p_X(x)\left\{\sum_{x_1'} p_{U_1|X_1}(u_1|x_1')(1 + \varepsilon L(u_1))\right\}}\right) \\
&= \max_{x_1, u_1} \log\left(\frac{p_{X_1,U_1}(x_1, u_1)|\mathcal{X}_1|}{p_X(x)\left\{\sum_{x_1'} p_{U_1|X_1}(u_1|x_1')\right\}}\right) \\
&= I_{\max}(X_1; U_1)_p \ .
\end{aligned}$$

We also get from (89) $p_\varepsilon(x_2, u_2) = p(x_2, u_2)$ and thus $I_{\max}(X_2; U_2)_{p_\varepsilon} = I_{\max}(X_2; U_2)_p$. Note that from Fact 5 (91) is automatically satisfied. Now, we choose ε such that $\min_{u_1}(1 + \varepsilon L(u_1)) = 0$ (such an ε always exists since ε can be negative) and let $u_1 = u_1^*$ attain this minimum. This implies $p_\varepsilon(u_1^*) = 0$, and hence we can reduce the cardinality of $\mathcal{U}_1$ by 1 or equivalently there exists a U_1' such that $|\mathcal{U}_1'| \leq |\mathcal{U}_1| - 1$ and the minimum of $\nu_1 I_{\max}(X_1; U_1) + \nu_2 I(X_2; U_2)$ is preserved. Finally, we can proceed by induction until $|\mathcal{U}_1| = |\mathcal{X}_1||\mathcal{X}_2||\mathcal{Y}|$, beyond which we are no longer guaranteed the existence of a non-trivial perturbation $L(u_1)$ satisfying (90). Hence we can restrict the cardinality to $|\mathcal{U}_1| \leq |\mathcal{X}_1||\mathcal{X}_2||\mathcal{Y}|$. A very similar argument can be carried out for bounding the cardinality $|\mathcal{U}_2| \leq |\mathcal{X}_1||\mathcal{X}_2||\mathcal{Y}|$.
The same analysis can be carried out for the rate region of Lemma 1.2 and Lemma 2.2, where the rates are governed by smoothed max-mutual information $I^{\varepsilon}_{\max}(X_j; U_j)_p$ and the channel smoothed max-mutual information can be described as $I^{\varepsilon_j}_{\max}(p_{U_j|X_j}) = I^{\varepsilon_j}_{\max}(X_j; U_j)_{p_{X_j} p_{U_j|X_j}}$, since $I^{\varepsilon_j}_{\max}(p_{U_j|X_j})$ is independent of the choice of input. We can repeat the entire argument by replacing the distribution p with p', such that $p'_{X_j,U_j} \in \mathcal{B}^{\varepsilon}(p_{X_j,U_j})$ and $P'_{X_j} = p_{X_j}$, without disturbing the global Markov property. The only change is that we will start with the Lyapunov perturbation of the distribution $p_{X_1} p'_{U_1|X_1} p_{X_2} p'_{U_2|X_2} p_{Y|U_1,U_2}$ instead of $p_{X_1} p_{U_1|X_1} p_{X_2} p_{U_2|X_2} p_{Y|U_1,U_2}$. Then, one can run the same argument by first considering $p'_\gamma(u_1|x_1) = (1 + \gamma L(u_1))$ and showing that $I^{\varepsilon}_{\max}(X_1; U_1)_{q_{X_1} p'_\gamma(u_1|x_1)} = I^{\varepsilon}_{\max}(X_1; U_1)_p$. Note that under this perturbation $I^{\varepsilon}_{\max}(X_2; U_2)_p$ remains unchanged. Then the similar analysis can be done by considering the perturbation of $q_{X_2} p'_{U_2|X_2}(u_2|x_2) \in \mathcal{B}^{\varepsilon}(q_{X_2} p_{U_2|X_2}(u_2|x_2))$, by keeping q_{X_2} fixed. This shows that it suffices to choose U_1, U_2 with cardinality $|\mathcal{U}_j| \leq |\mathcal{X}_1||\mathcal{X}_2||\mathcal{Y}|$ and the region $\mathcal{R}_{outer}$ stays the same. ■

Remark 3.2: Another technique to prove cardinality bounds of auxiliary random variables in classical information theory is *support lemma* (a corollary of Fenchel-Eggleston-Carathéodory's theorem) [24, Appendix C] and recently a fully quantum version of it was developed for $I_{\max}$ by [7]. However, it does not suffice for our purpose. The reason being that the global Markov chain $(X_1, X_2) \to (U_1, U_2) \to Y$ cannot be preserved using support lemma. Hence, we have to use the perturbation technique as in the proof above. It is for this reason, the cardinaility bounds for the asymptotic iid case as shown in [5, Lemma 5] are also proven using the perturbation method.

C. Cardinality of $\mathcal{U}_1, \mathcal{U}_2$ for CS-QC MAC simulation

Proof of Lemma 3.3:

Proof: The proof is very similar to that of the proof of Lemma 1.3 since the auxiliary random variables U_1, U_2 are classical and are generated conditioned on X_j, the output of the measurement operators of the channel. We have:

$$I_{\max}(E_j; U_j)_\tau = \inf_{r^{U_j}} \left\| \left(\tau^{E_j} \otimes r^{U_j}\right)^{-1/2} \tau^{E_j U_j} \left(\tau^{E_j} \otimes r^{U_j}\right)^{-1/2} \right\|_\infty , \tag{92}$$

where $\tau^{E_jU_j} := \sum_{x_j,u_j} p_{X_j}(x_j)\left[p_{U_j|X_j}(u_j|x_j)\,|u_j\rangle\langle u_j|^{U_j}\right] \otimes \varphi_{x_j}^{E_j}$. Since r^{U_j} is classical it can be chosen to be of form:

$$r^{U_j} = \sum_{u_j} r_{U_j}(u_j)\,|u_j\rangle\langle u_j|^{U_j} \ ,$$

for any probability distribution r_{U_j}. Substituting this and $\tau^{E_j} = \sum_{x_j} p_{X_j}(x_j)\varphi_{x_j}^{E_j}$ in (92) we get:

$$\begin{aligned} &I_{\max}(E_j;U_j)_\tau \\ &= \inf_{r_{U_j}} \left\| \sum_{x_j,u_j} \left[\left(\sum_{x_j'} p_{X_j}(x_j')\varphi_{x_j'} \right)^{-1/2} \varphi_{x_j} \left(\sum_{x_j}^{''} p_{X_j}(x_j'')\varphi_{x_j''} \right)^{-1/2} \right] \otimes \frac{p_{X_j}(x_j)p_{U_j|X_j}(u_j|x_j)}{r_{U_j}(u_j)} |u_j\rangle\langle u_j| \right\|_\infty \\ &\overset{(a)}{=} \left\| \sum_{x_j,u_j} \left[\left(\sum_{x_j'} p_{X_j}(x_j')\varphi_{x_j'} \right)^{-1/2} \varphi_{x_j} \left(\sum_{x_j}^{''} p_{X_j}(x_j'')\varphi_{x_j''} \right)^{-1/2} \right] \otimes \frac{p_{X_j}(x_j)|\mathcal{X}_j| p_{U_j|X_j}(u_j|x_j)}{\sum_{x_j'} p_{U_j|X_j}(u_j|x_j')} |u_j\rangle\langle u_j| \right\|_\infty , \end{aligned} \tag{93}$$

where (a) follows because U_j is classical and only depends on classical X_j and consequently from Fact 5. With this identification we can essentially repeat the proof of Lemma 1.3 given in Section B-B. In order to do this we can perturb only the distribution $p_{U_j|X_j}$ in the state τ defined in (49), that is used to evaluate $\mathcal{R}_{outer}^{QC-fb}$. Since the quantity $I_{\max}(E_j;U_j)_{\tau_j}$ depends on U_j via $p_{U_j|X_j}$, it follows exactly from the proof in Section B-B that $I_{\max}(E_j;U_j)$ remains the same under perturbation. The rest of the proof is exactly the same as that in Section B-B.

■

D. *Single letterization of asymptotic expansion for MAC simulation*

Here, we prove a single-letter characterization of $\mathcal{R}_{outer}$. Although, this is a well known technique in classical information theory, yet an argument is always needed for the task under consideration. Hence, we give a self contained proof to show that our one-shot rate region can be lifted to the asymptotic iid setting and also can be single-letterized to match the outer bound of [5, Theorem 4].

Proposition 1: Consider any n-letter simulation code that induces the joint distribution

$$p'_{X_1^n,U_1^n,X_2^n,U_2^n,Y^n}(x_1^n,u_1^n,x_2^n,u_2^n,y^n) := q_{X_1}^{\otimes n}(x_1)q_{X_2}^{\otimes n}(x_2)p'_{U_1^n|X_1^n}(u_1^n|x_1^n)p'_{U_2^n|X_2^n}(u_2^n|x_2^n)p'_{Y^n|U_1^n,U_2^n}(y^n|u_1^n,u_2^n), \tag{94}$$

such that $p'_{X_1^n,X_2^n,Y^n} = q_{X_1}^{\otimes n}q_{X_2}^{\otimes n}q_{Y|X_1,X_2}^{\otimes n}$. Suppose the rate pair (R_1,R_2) satisfy:

$$R_j \geq \frac{1}{n} I(X_j^n, U_j^n)_{p'}, \text{ for } j \in \{1,2\}.$$

Then,

$$R_j \geq I(X_j;U_j)_p \text{ for } j \in \{1,2\} \ ,$$

for some distribution $p_{X_1,U_1,X_2,U_2,Y} := q_{X_1}q_{X_2}p_{U_1|X_1}p_{U_2|X_2}p_{Y|U_1,U_2}$ such that $p_{X_1,X_2,Y} = q_{X_1}q_{X_2}q_{Y|X_1,X_2}$.

Proof: The proof follows mostly by standard arguments. An important point is to ensure that the cardinalities of the auxiliaries (U_1,U_2) are bounded, which is ascertained by Fact 4.

The proof of single-letterization is as follows (for $j \in \{1,2\}$):

$$\begin{aligned} R_j &\geq \frac{1}{n} I(X_j^n;U_j^n)_{p'} \\ &= \frac{1}{n}\left[H(X_j^n)_{q_{X_j}^{\otimes n}} - H(X_j^n|U_j^n)_{p'} \right] \\ &\overset{(a)}{=} \frac{1}{n}\left[\sum_{i=1}^n H(X_{j,i})_{q_{X_j}} - H(X_j^n|U_j^n)_{p'} \right] \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} \frac{1}{n} \left[H(X_{j,i})_{q_{X_j}} - H(X_{j,i}^n | X_{j,1}^{i-1}, U_j^n)_{p'} \right] \\
&\overset{(b)}{\geq} \sum_{i=1}^{n} \frac{1}{n} \left[H(X_{j,i})_{q_{X_j}} - H(X_{j,i} | Uj, i)_{p'} \right] \\
&= \sum_{i=1}^{n} \frac{1}{n} I(X_{j,i}; U_{j,i})_{p'} \\
&\overset{(c)}{=} \sum_{i=1}^{n} \frac{1}{n} I(X_{j,i}; U_{j,i} | T = i)_{p'} \\
&= I(X_{j,T}; U_{j,T} | T)_{p'} \\
&\overset{(d)}{=} I(X_j; U_{j,T}, T)_{\pi_T p'},
\end{aligned}
\tag{95}
$$

where (a) follows by iid distribution of X_j; (b) holds by the fact that conditioning reduces entropy; (c) follows by identifying the so-called time-sharing random variable T, uniformly distributed on $\{1, 2, \ldots, n\}$ with p.m.f. $\pi_T(i) = \frac{1}{n}$ and independent of X_1, X_2, U_1, U_2, Y; (d) holds because $X_{j,T} \perp\!\!\!\perp T$ and with the identification $X_{j,T} = X_j$.

Now we define $U_j := (U_{j,T}, T)$, $Y_j := Y_{j,T}$ and the joint distribution

$$
p_{X_1,X_2,U_1,U_2,Y} := \sum_{i=1}^{n} \frac{1}{n} p'_{X_{1,i},X_{2,i}U_{1,i},U_{2,1},Y|T=i} .
$$

The above defined p satisfy the marginal property (or the simulation constraint):

$$
\begin{aligned}
\sum_{u_1,u_2} p(x_1, x_2, u_1, u_2, y) &= \sum_{u_1,u_2} \sum_{i=1}^{n} \frac{1}{n} p'_{X_{1,i},X_{2,i}U_{1,i},U_{2,1},Y_i|T}(x_1, x_2, u_1, u_2, y | T = i) \\
&= \sum_{i=1}^{n} \frac{1}{n} \sum_{u_1,u_2} p'_{X_{1,i},X_{2,i}U_{1,i},U_{2,1},Y_i}(x_1, x_2, u_1, u_2, y) \\
&\overset{(i)}{=} \sum_{i=1}^{n} \frac{1}{n} q_{X_1}(x_1) q_{X_2}(x_2) q_{Y|X_1X_2}(y | x_1, x_2) \\
&= q_{X_1}(x_1) q_{X_2}(x_2) q_{Y|X_1X_2}(y | x_1, x_2),
\end{aligned}
$$

where (i) follows from (94).
Finally, from Fact 4, we have that it suffices to take U_j with cardinality $|\mathcal{U}_j| \leq |\mathcal{X}_1||\mathcal{X}_2||\mathcal{Y}|$. We have thus identified p that satisfies the simulation constraint and using this p in (95), we obtain

$$
R_j \geq I(X_j; U_j)_p .
$$

■

E. Single letterization for asymptotic expansion of CS-QC MAC simulation

We now state and prove a proposition that ensures that CS-QC MAC simulation with feedback has a single-letter characterization.

Proposition 2: Consider any n-letter simulation code, that induces the state

$$
\begin{aligned}
\tau'^{E_1^n X_1^n U_1^n E_2^n X_2^n U_2^n Y^n} := \sum_{\substack{x_1^n, x_2^n \\ u_1^n, u_2^n, y^n}} & p_{X_1}^{\otimes n}(x_1) p_{X_2}^{\otimes n}(x_2) p'_{U_1^n|X_1^n}(u_1^n | x_1^n) p'_{U_2^n|X_2^n}(u_2^n | x_2^n) p'_{Y^n|U_1^n,U_2^n}(y^n | u_1^n, u_2^n) \, |x_1^n\rangle\langle x_1^n|_{X_1^n} \\
& |u_1^n\rangle\langle u_1^n|_{U_1^n} \otimes |x_2^n\rangle\langle x_2^n|_{X_2^n} \otimes |u_2^n\rangle\langle u_2^n|_{U_2^n} \otimes \left(\varphi_{x_1}^{E_1}\right)^{\otimes n} \otimes \left(\varphi_{x_2}^{E_2}\right)^{\otimes n} \otimes |y^n\rangle\langle y^n|_{Y_1^n} ,
\end{aligned}
\tag{96}
$$

such that $\tau'^{E_1^n E_2^n Y^n} = \eta_{E_1 E_2 Y}^{\otimes n}$. Suppose the rate pair (R_1, R_2) satisfy:

$$R_j \geq \frac{1}{n} I(E_j^n, U_j^n)_{\tau'}, \text{ for } j \in \{1,2\}.$$

Then,

$$R_j \geq I(E_j; U_j)_{\tau} \text{ for } j \in \{1,2\}\ ,$$

for some state

$$\tau^{E_1 X_1 U_1 E_2 X_2 U_2 Y} := \sum_{\substack{x_1,x_2\\u_1,u_2,y}} p_{X_1}(x_1) p_{X_2}(x_2) p_{U_1|X_1}(u_1|x_1) p_{U_2|X_2}(u_2|x_2) p_{Y|U_1,U_2}(y|u_1,u_2) \varphi_{x_1}^{E_1} \otimes \varphi_{x_2}^{E_2} \otimes$$
$$|x_1\rangle\langle x_1|^{X_1} \otimes |u_1\rangle\langle u_1|^{U_1} \otimes |x_2\rangle\langle x_2|^{X_2} \otimes |u_2\rangle\langle u_2|^{U_2} \otimes |y\rangle\langle y|^{Y}\ , \tag{97}$$

such that $\tau^{E_1 E_2 Y} = \eta^{E_1 E_2 Y}$.

Proof: The proof is very similar to that of Proposition 1. By Lemma 3.3, we can assume that cardinalities of $|\mathcal{U}_1|, |\mathcal{U}_2|$ are bounded as $|\mathcal{U}_j| \leq |\mathcal{X}_1||\mathcal{X}_2||\mathcal{Y}|$. Then, we have

$$\begin{aligned}
R_j &\geq \frac{1}{n} I(E_j^n; U_j^n)_{\tau'} \\
&= \frac{1}{n}\left[H(E_j^n)_{\tau'^{\otimes n}_{E_j}} - H(E_j^n|U_j^n)_{\tau'}\right] \\
&\overset{(a)}{=} \frac{1}{n}\left[\sum_{i=1}^n H(E_{j,i})_{\tau'_{E_{j,i}}} - H(E_j^n|U_j^n)_{\tau'}\right] \\
&= \sum_{i=1}^n \frac{1}{n}\left[H(E_{j,i})_{\tau'_{E_{j,i}}} - H(E_{j,i}|E_{j,1}^{i-1}, U_j^n)_{\tau'}\right] \\
&\overset{(b)}{\geq} \sum_{i=1}^n \frac{1}{n}\left[H(E_{j,i})_{\tau'_{E_{j,i}}} - H(E_{j,i}|U_{j,i})_{\tau'}\right] \\
&= \sum_{i=1}^n \frac{1}{n} I(E_{j,i}; U_{j,i})_{\tau'} \\
&\overset{(c)}{=} \sum_{i=1}^n \frac{1}{n} I(E_{j,i}; U_{j,i}|T=i)_{\tau'} \\
&= I(E_{j,T}; U_{j,T}|T)_{\tau'} \\
&\overset{(d)}{=} I(E_j; U_{j,T}, T)_{\pi_T \tau'},
\end{aligned} \tag{98}$$

where (a) follows by iid distribution of E_j; (b) holds by the fact that conditioning reduces entropy; (c) follows by identifying the so-called time-sharing random variable T, uniformly distributed on $\{1, 2, \ldots, n\}$ with p.m.f. $\pi_T(i) = \frac{1}{n}$, and independent of E_1, E_2, U_1, U_2, Y; (d) holds because $E_j \perp\!\!\!\perp T$ and with the identification $E_{j,T} = E_j$.

Now we define $U_j := (U_{j,T}, T)$, $X_j := X_{j,T}$, $Y_j := Y_{j,T}$ $(X_{j,T}, Y_{j,T} \perp\!\!\!\perp T)$ and the overall state

$$\tau^{E_1 X_1 U_1 E_2 U_2 Y} := \sum_{i=1}^n \frac{1}{n} \sum_{\substack{x_{1,i},x_{2,i}\\u_{2,i},u_{1,i},y}} p'_{X_{1,i}|T=i}(x_{1,i}) p'_{X_{2,i}|T=i}(x_{2,i}) p'_{U_{1,i}|X_{1,i},T=i}(u_{1,i}|x_{1,i}) p'_{U_{2,i}|X_{2,i},T=i}(u_{2,i}|x_{2,i})$$
$$p'_{Y|U_{1,i},U_{2,i},T=i}(y|u_{1,i}u_{2,i})\, |x_{1,i}\rangle\langle x_{1,i}|^{X_{1,i}} \otimes |x_{2,i}\rangle\langle x_{2,i}|^{X_{2,i}}\, \varphi_{x_{1,i}}^{E_1} \otimes \varphi_{x_{2,i}}^{E_2} \otimes |u_{1,i}\rangle\langle u_{1,i}|^{U_1} \otimes |u_{2,i}\rangle\langle u_{2,i}|^{U_2} \otimes |y\rangle\langle y|^{Y}\ .$$

The above defined τ satisfies the simulation constraint since:

$$\tau^{E_1 E_2 X_1 X_2 Y} = \mathrm{Tr}\left[\sum_{\substack{x_1,x_2\\u_1,u_2,y}} \sum_{i=1}^n \frac{1}{n} p'_{X_{1,i},X_{2,i}U_{1,i},U_{2,i},Y_i|T}(x_1,x_2,u_1,u_2,y|T=i)\, |x_1\rangle\langle x_1|^{X_1} \otimes |x_2\rangle\langle x_2|^{X_2} \otimes\right.$$

$$\varphi_{x_1}^{E_1} \otimes \varphi_{x_2}^{E_2} \otimes |u_{1,i}\rangle\langle u_{1,i}|^{U_{1,i}} \otimes |u_{2,i}\rangle\langle u_{2,i}|^{U_{2,i}} \otimes |y_i\rangle\langle y_i|^Y \Big]$$

$$\overset{(i)}{=} \mathrm{Tr}\left[\sum_{\substack{x_1,x_2\\u_1,u_2,y}} p_{X_1,X_2,U_1,U_2,Y}(x_1,x_2,u_1,u_2,y)\,|x_1\rangle\langle x_1|^{X_1} \otimes |x_2\rangle\langle x_2|^{X_2} \otimes \varphi_{x_1}^{E_1} \otimes \varphi_{x_2}^{E_2} \otimes \right.$$

$$\left. |u_1\rangle\langle u_1|^{U_1} \otimes |u_2\rangle\langle u_2|^{U_2} \otimes |y\rangle\langle y|^Y \right]$$

$$\overset{(ii)}{=} \mathrm{Tr}\left[\sum_{\substack{x_1,x_2\\u_1,u_2,y}} p_{X_1}(x_1)p_{X_2}(x_2)p_{U_1|X_1}(u_1|x_1)p_{U_2|X_2}(u_2|x_2)p_{Y|U_1,U_2}(y|u_1,u_2)\,|x_1\rangle\langle x_1|^{X_1} \otimes |x_2\rangle\langle x_2|^{X_2}\right.$$

$$\left. \otimes\varphi_{x_1}^{E_1} \otimes \varphi_{x_2}^{E_2} \otimes p_{X_2}(x_2)\varphi_{x_2}^{E_2} \otimes q_{Y|X_1X_2}(y|x_1,x_2)\,|y\rangle\langle y|^Y \right]$$

$$= \eta^{E_1E_2X_1X_2Y}, \tag{99}$$

where (i) holds by defining

$$p_{X_1,X_2U_1,U_2,Y}(x_1,x_2,u_1,u_2,y) := \sum_{i=1}^{n} \frac{1}{n} \sum_{u_1,u_2} p'_{X_{1,i},X_{2,i}U_{1,i},U_{2,1},Y_i}(x_1,x_2,u_1,u_2,y)$$
$$= p_{X_1}(x_1)p_{X_2}(x_2)p_{U_1|X_1}(u_1|x_1)p_{U_2|X_2}(u_2|x_2)p_{Y|U_1,U_2}(y|u_1,u_2),$$

and (ii) follows from (96) and the identification of p in (i) above.
We have thus identified τ that satisfies the simulation constraint and using this τ in (98), we obtain

$$R_j \geq I(E_j;U_j)_\tau \ .$$

■

F. Continuity of the ε-rate region

We start with the following fact about the continuity of mutual information $I(A;B)_{\tau^{AB}}$ with respect to $\tau_{A,B}$ having a fixed marginal τ^A. When both the systems A and B are classical, the continuity bounds of [25, Theorem 17.3.3] and [14, Lemma 4]) suffice. However, an improved version of the continuity of mutual information for general quantum states was proven in [26, Lemma 2] known as the Alicki-Fannes-Winter continuity bound, which also applies to classical bipartite distributions. We state this result as the following fact:

Fact 6: [26, Lemma 2] Let ρ'^{AB} and ρ^{AB} be two quantum states on the joint Hilbert space $\mathcal{H}^{AB}$ such that $\rho^A = \rho'^A$. Then for any $\varepsilon \in (0,1)$, it holds that:

$$\left\|\rho^{AB} - \rho'^{AB}\right\|_{tvd} \leq \varepsilon \Rightarrow |I(A;B)_\rho - I(A;B)_{\rho'}| \leq 2\varepsilon \log|\mathcal{A}| + 2h_2\left(\frac{\varepsilon}{1+\varepsilon}\right).$$

We now state and prove that the asymptotic extension of our outer bound, that is, $\mathcal{R}^{iid}_{outer}(\varepsilon)$ converges to $\mathcal{R}^{iid}$ as mentioned in Corollary 1.1. Since this is a direct consequence of [14, Lemma 6], we state it as the following fact and include the similar proof for completeness.

Fact 7: [14, Lemma 6] Consider the asymptotic iid setting for simulating a MAC $q_{Y|X_1,X_2}$ with inputs $q_{\vec{X}} = q_{X_1} \times q_{X_2}$ (see Definition 5) for any $\varepsilon \in (0,1)$. Let an outer bound for this task be given by:

$$\mathcal{R}^{iid}_{outer}(\varepsilon) := \left\{(R_1,R_2) : R_j \geq I(X_j;U_j)_p, \text{ for } j \in \{1,2\},\ p_{\vec{X},\vec{U},Y}\ s.t.\ \mathop{\mathbb{E}}_{q_{\vec{X}}} \left\|p_{Y|\vec{X}} - q_{Y|\vec{X}}\right\|_{tvd} \leq \varepsilon\right\}$$

with $|\mathcal{U}_j| \leq |\mathcal{X}_1||\mathcal{X}_2||\mathcal{Y}|$ for $j \in \{1,2\}$ then:

$$\mathcal{R}^{iid} = \bigcap_{\varepsilon>0} \mathcal{R}^{iid}_{outer}(\varepsilon),$$

where $\mathcal{R}^{iid} := \left\{ (R_1, R_2) : R_j \geq I(X_j; U_j)_p, \text{ for } j \in \{1,2\}, \; p_{\vec{X},\vec{U},Y} \; s.t. \; p_{\vec{X},Y} = q_{\vec{X},Y} \right\}$.

Proof: Since any rate pair $(R_1, R_2) \in \mathcal{R}^{iid}$ lies in $\mathcal{R}^{iid}_{\varepsilon}$ for all $\varepsilon > 0$, therefore it holds that $\mathcal{R}^{iid} \subseteq \bigcap_{\varepsilon>0} \mathcal{R}^{iid}_{outer}(\varepsilon)$. The reverse direction $\bigcap_{\varepsilon>0} \mathcal{R}^{iid}_{outer}(\varepsilon) \subseteq \mathcal{R}^{iid}$ is quite non-trivial, for which we sketch the steps below. Consider a sequence $\{\varepsilon_i\}_{i\geq 1}$ such that $\lim_{i\to\infty} \varepsilon_i = 0$. Also, let

$$\mathcal{P}(\vec{r}) := \left\{ p_{\vec{X},\vec{U},Y} : q_{X_1} q_{X_2} p_{U_1|X_1} p_{U_2|X_2} p_{Y|U_1,U_2} \text{ and } p_{\vec{X},Y} = q_{\vec{X},Y} \text{ with } \{|\mathcal{U}_j|\}_{j=1}^2 \leq |\mathcal{X}_1||\mathcal{X}_2||\mathcal{Y}| \right\} .$$
$$\mathcal{P}_{\varepsilon}(\vec{r}) := \left\{ p_{\vec{X},\vec{U},Y} : q_{X_1} q_{X_2} p_{U_1|X_1} p_{U_2|X_2} p_{Y|U_1,U_2} \text{ and } \left\| p_{\vec{X},Y} - q_{\vec{X},Y} \right\|_{tvd} \leq \varepsilon \text{ with } \{|\mathcal{U}_j|\}_{j=1}^2 \leq |\mathcal{X}_1||\mathcal{X}_2||\mathcal{Y}| \right\} .$$

Now, we take a rate pair, say $\vec{R}^* = (R_1^*, R_2^*) \in \bigcap_{\varepsilon>0} \mathcal{R}^{iid}_{outer}(\varepsilon)$. There is a sequence of p.m.f. $p_i(\vec{x}, \vec{u}, y) \in \mathcal{P}_{\varepsilon_i}(\vec{r})$ corresponding to this rate pair. These p.m.f.s belong to the probability simplex $\mathcal{P}$ of dimension $|\mathcal{X}_1||\mathcal{X}_2||\mathcal{U}_1||\mathcal{U}_2||\mathcal{Y}|$ and since the cardinalities of $|\mathcal{U}_1|, |\mathcal{U}_2|$ are bounded therefore the probability simplex is compact. Thus, there exists a subsequence $\{i_k\}_{k\geq 1}$ such that the subsequence of p.m.f. $\{p_{i_k}(\vec{x}, \vec{u}, y)\}_{k\geq 1}$ converges to some $p^*(\vec{x}, \vec{u}, y)$ in the probability simplex.
The key point of the proof is that $p^*(\vec{x}, \vec{u}, y) \in \mathcal{P}(\vec{r})$. This can be proven by using the continuity of the total variation distance and the mutual information in the probability simplex. In particular, this follows from

$$\|p^*(\vec{x}, y) - q(\vec{x}, y)\|_1 = \lim_{k\to\infty} \|p_{i_k}(\vec{x}, y) - q(\vec{x}, y)\|_1 = 0 \Rightarrow p^*(\vec{x}, y) = q(\vec{x}, y)$$

Furthermore, since $(X_{1i}, U_{1i}) \perp\!\!\!\perp (X_{2i}, U_{2i})$ and the Markov condition $(X_{1i}, X_{2i}) \to (U_{1i}, U_{2i}) \to Y_i$ holds for all $i \geq 1$, the same also holds in the limiting case.
Finally it can also be shown that $\vec{R}^*$ is a point of $\mathcal{R}^{iid}$ corresponding to the p.m.f. $p^*(\vec{x}, \vec{u}, y)$. By using Fact 6, we get:

$$\left\| p^*_{X_j,U_j} - p_{X_j,U_j} \right\|_{tvd} \leq \varepsilon \Rightarrow \left| I(X_j; U_j)_{p^*_{X_j,U_j}} - I(X_j; U_j)_{p_{X_j,U_j}} \right| \leq 2\varepsilon \log(|\mathcal{X}_j|) - 2h_2\left(\frac{\varepsilon}{1+\varepsilon}\right) := g(\varepsilon).$$

This implies that $\lim_{\varepsilon\to 0} g(\varepsilon) = 0$. Hence, $\mathcal{R}^{iid}_{outer}(\varepsilon_{i_k})$ converges to $\mathcal{R}^{iid}$. This completes the proof. ■

APPENDIX C
ASYMPTOTIC EQUIPARTITION PROPERTY (AEP)

A. Asymptotic expansion for fixed input

We state the AEP for smoothed max-mutual information for a classical distribution (see (34) following Theorem 9 of [10]) and a CQ state (equation (107) following Theorem 11 of [10]).

Fact 8: [10, Equations (34),(107)] Let $q_X p_{X|U} = p_{X,U} \in \mathcal{P}$ be any joint distribution and $\tau^{EU} \in \mathcal{D}(\mathcal{H}^{EU})$ be any quantum state. Then for any $\varepsilon \in (0, 1)$, it holds that

$$\lim_{n\to\infty} \frac{1}{n} I^{\varepsilon}_{\max}(X : U)_{p^{\otimes n}_{X,U}} = I(X : U)_{p_{X,U}}, \text{and} \tag{100}$$
$$\lim_{n\to\infty} \frac{1}{n} I^{\varepsilon}_{\max}(E : U)_{(\tau^{E,U})^{\otimes n}} = I(E : U)_{\tau^{E,U}}. \tag{101}$$

B. Asymptotic expansion for universal simulation

We now state the AEP for smoothed max-mutual information of the channel. This is obtained by taking the limit $\lim_{n\to\infty}$ in [11, Corollary 9] resulting in the following fact:

Fact 9: [11, Corollary 9] For any point-to-point channel $p_{U|X}$ and $\varepsilon \in (0, 1)$, it holds that

$$\lim_{n\to\infty} \frac{1}{n} I^{\varepsilon}_{\max}(p^{\otimes n}_{U|X}) = \max_{q_X} I(X; U)_{q_X p_{U|X}}.$$

APPENDIX D
UNIVERSAL MAC SIMULATION

A. Proof of Lemma 2.1

We start by giving the achievability proof of Lemma 2.1.

Proof: Fix $(\varepsilon_1, \varepsilon_2, \delta)$ satisfying the conditions of Lemma 2.1 and choose auxiliary channels $p_{U_1|X_1}, p_{U_2|X_2}$ and the decoder $p_{Y|U_1,U_2}$ from the set $\mathcal{A}^{inner}$, given in (39). We need to show that for any $(R_1, R_2) \in \mathcal{R}_{\mathbb{U}}^{inner}(\varepsilon_1, \varepsilon_2, \delta)$ (defined in (38)), there exists an (R_1, R_2, ε) one-shot universal MAC simulation protocol as mentioned in Definition 8.

We will use the universal point-to-point channel simulation algorithm of Fact 2-(ii) to simulate the auxiliary channels $p_{U_j|X_j}$ independently at each sender $j \in \{1, 2\}$.

- **Sender-**j: Let s_{U_j} be a distribution with full support and choose $U_j \sim s_{U_j}$ as the shared randomness between the pair $(\mathcal{E}_j, \mathcal{D})$. Using the rejection sampling algorithm mentioned in Fact 1, sender j sends the appropriately chosen index of the shared randomness using R_j bits to simulate the auxiliary channel $p_{U_j|X_j}$, irrespective of any particular input q_{X_j}.
- **Decoding:** After receiving the transmitted index of shared randomness from both the encoders, the decoder first generates (U_1, U_2) and applies the stochastic map $Y \sim p_{Y|U_1,U_2}$ to universally simulate $q_{Y|X_1,X_2}$.
- Hence, the output distribution of U_j at $\mathcal{D}$, denoted by $p^{algo}_{U_j|X_j}$ satisfies (from Fact 2-(ii)):

$$\max_{x_j} \left\| p^{algo}_{U_j|X_j=x_j} - p_{U_j|X_j=x_j} \right\|_{tvd} \leq \varepsilon_j \ . \tag{102}$$

  The amount of classical communication required to achieve this target distribution is given by the universal point-to-point channel simulation protocol from Fact 2-(ii) as :

$$R_j \geq I_{\max}^{\varepsilon_j - \delta}(p_{U_j|X_j}) + \log\log\frac{1}{\delta} \ .$$

Thus, our algorithm generates the overall distribution

$$p^{algo}_{X_1,X_2,U_1,U_2,Y} = q_{X_1} \times q_{X_2} \times p^{algo}_{U_1|X_1} \times p^{algo}_{U_2|X_2} p_{Y|U_1,U_2}. \tag{103}$$

Note that the input distributions $q_{X_j}(x_j)$ above are arbitrary and play no role in the simulation criteria given by (36).
To complete the proof, we finally need to show

$$\max_{x_1,x_2} \| p^{algo}_{Y|X_1=x_1,X_2=x_2} - q_{Y|X_1=x_1,X_2=x_2} \|_{tvd} \leq \varepsilon_1 + \varepsilon_2.$$

This follows by the following chain of inequalities:

$$\begin{aligned}
&\max_{x_1,x_2} \| p^{\text{algo}}_{Y|X_1=x_1,X_2=x_2} - q_{Y|X_1=x_1,X_2=x_2} \|_{tvd} \\
&\overset{(a)}{\leq} \max_{x_1,x_2} \| p^{\text{algo}}_{Y|X_1=x_1,X_2=x_2} - p_{Y|X_1=x_1,X_2=x_2} \|_{tvd} + \max_{x_1,x_2} \| p_{Y|X_1=x_1,X_2=x_2} - q_{Y|X_1=x_1,X_2=x_2} \|_{tvd} \\
&\overset{(b)}{=} \max_{x_1,x_2} \left\| \sum_{u_1,u_2} p_{Y|U_1=u_1,U_2=u_2} \left(p^{\text{algo}}_{U_1|X_1}(u_1) p^{\text{algo}}_{U_2|X_2}(u_2) - p_{U_1|X_1}(u_1|x_1) p_{U_2|X_2}(u_2|x_2) \right) \right\|_{tvd} \\
&= \max_{x_1,x_2} \sum_{u_1,u_2} \sum_{y} p_{Y|U_1,U_2}(y|u_1,u_2) \left| \left(p^{\text{algo}}_{U_1|X_1}(u_1) p^{\text{algo}}_{U_2|X_2}(u_2) - p_{U_1|X_1}(u_1|x_1) p_{U_2|X_2}(u_2|x_2) \right) \right| \\
&\overset{(c)}{\leq} \max_{x_1,x_2} \| p^{\text{algo}}_{U_1|X_1=x_1} p^{algo}_{U_2|X_2=x_2} - p^{algo}_{U_1|X_1=x_1} p_{U_2|X_2=x_2} \|_{tvd} + \max_{x_1,x_2} \| p^{\text{algo}}_{U_1|X_1=x_1} p_{U_2|X_2=x_2} - p_{U_1|X_1=x_1} p_{U_2|X_2=x_2} \|_{tvd} \\
&= \max_{x_1} \| p^{\text{algo}}_{U_1|X_1=x_1} \|_1 \max_{x_2} \| p^{algo}_{U_2|X_2=x_2} - p_{U_2|X_2=x_2} \|_{tvd} + \max_{x_2} \| p_{U_2|X_2=x_2} \|_1 \max_{x_1} \| p^{\text{algo}}_{U_1|X_1=x_1} - p_{U_1|X_1=x_1} \|_{tvd} \\
&\overset{(d)}{\leq} \varepsilon_1 + \varepsilon_2
\end{aligned}$$

where (a) and (c) follow from the triangle inequality and the fact that maximum value of the sum of two non-negative functions is less than or equal to the sum of their individual maximum values; (b) follows from the definition of distribution induced by the code in (103) and (d) follows from (102).
We have thus shown that there exists an $(R_1, R_2, \varepsilon_1+\varepsilon_2)$ code for universally simulating $q_{Y|X_1,X_2}$ which implies $\mathcal{R}_{\mathbb{U}}^{inner}(\varepsilon_1,\varepsilon_2) \subseteq \mathcal{R}_{\mathbb{U}}$. ■

B. Proof of Lemma 2.2

We now prove Lemma 2.2, the converse for the one-shot universal MAC simulation task.

Proof: Let $(\varepsilon_1, \varepsilon_2)$ and ε satisfy the conditions of the lemma. We need to show that any (R_1, R_2, ε) MAC simulation protocol according to Definition 8 has $(R_1, R_2) \in \mathcal{R}_{\mathbb{U}}^{outer}(\varepsilon_1,\varepsilon_2)$ (defined in (40)), which implies $\mathcal{R}_{\mathbb{U}} \subseteq \mathcal{R}_{\mathbb{U}}^{outer}(\varepsilon_1,\varepsilon_2)$.

Consider a MAC simulation protocol with any arbitrary input $q_{X_1} \times q_{X_2}$ and the overall distribution as

$$\bigotimes_{j=1}^{2} \left(q_{X_j} q_{S_j} p'_{M_j|S_j,X_j} \right) p'_{Y|\vec{M},\vec{S}}.$$

The encoders are specified by ${p'}_{M_1|X_1,S_1}$ and ${p'}_{M_2|X_2,S_2}$, and the decoder is specified by ${p'}_{Y|M_1,M_2,S_1,S_2}$. Since, the code is a faithful simulation code, we have from Definition 8:

$$\max_{x_1,x_2} \left\| {p'}_{Y|X_1=x_1,X_2=x_2} - q_{Y|X_1=x_1,X_2=x_2} \right\|_{tvd} \leq \varepsilon = \varepsilon_1 + \varepsilon_2. \tag{104}$$

We now use a similar intuition as in Lemma 1.2 to identify the auxiliary random variables for simulating $q_{Y|X_1,X_2}$. Define the following set for every vector $\vec{x} = (x_1, x_2)$

$$\bar{\mathcal{C}}_{\vec{x}} := \left\{ (\vec{m}, \vec{s}) : p'_{M_j|S_j,X_j}(m_j|s_j, x_j) \geq \frac{\varepsilon_j}{|\mathcal{M}_j|}, j = 1, 2 \right\}. \tag{105}$$

We henceforth denote the projection of $\mathcal{C}_{\vec{x}}$ onto (M_j, S_j, X_j) (or the j^{th} user) as $\mathcal{C}_{x_j}$ and we make the similar identification for their respective complements.
Note that by union bound, we have

$$\mathbb{P}_{p'}(\mathcal{C}_{\vec{x}}) \leq \sum_{j=1}^{2} \mathbb{P}\left(\left\{ p'_{M_j|S_j,X_j}(m_j|s_j, x_j) \leq \frac{\varepsilon_j}{|\mathcal{M}_j|} \right\} \right) \leq \varepsilon_1 + \varepsilon_2, \tag{106}$$

where we have used:

$$\mathbb{P}_{p'}\left(\left\{ (m_j, s_j) : p'_{M_j|S_j,X_j}(m_j|s_j, x_j) \leq \frac{\varepsilon}{|\mathcal{M}_j|} \right\} \right) = \sum_{(m_j,s_j): p'_{M_j|S_j,X_j}(m_j|s_j,x_j) \leq \frac{\varepsilon_j}{|\mathcal{M}_j|}} p'_{S_j}(s_j) p'_{M_j|S_j,X_j}(m_j|s_j, x_j)$$

$$\Rightarrow \mathbb{P}_{p'}(\mathcal{C}_{x_j}) \leq \sum_{(m_j,s_j)} \frac{\varepsilon_j}{|\mathcal{M}_j|} q_{S_j} \leq \varepsilon_j. \tag{107}$$

Hence, $\mathbb{P}_{p'}(\bar{\mathcal{C}}_{\vec{x}}) \geq 1 - \varepsilon_1 - \varepsilon_2$, for all $\vec{x}$.
Consider the distribution defined as follows:

$$p_{M_j,S_j|X_j}(m_j, s_j|x_j) := \begin{cases} \frac{p'_{S_j}(s_j) p'_{M_j|S_j,X_j}(m_j|s_j,x_j)}{\mathbb{P}_{p'}(\bar{\mathcal{C}}_{x_j})}, & \text{if } m_j, s_j \in \bar{\mathcal{C}}_{x_j} \\ 0 & otherwise\ . \end{cases} \tag{108}$$

We have thus identified the auxiliary random variable $\{U_j\}_{j=1}^{2}$ for each x_j as:

$$U_j := (M_j, S_j)\mathbb{1}_{\bar{\mathcal{C}}_{x_j}} \equiv p_{U_j|X_j}(u_j|x_j) := p_{M_j,S_j|X_j}(m_j, s_j|x_j) = \frac{p'_{S_j}(s_j) p'_{M_j|S_j,X_j}(m_j|s_j, x_j) \mathbb{1}_{(m_j,s_j) \in \bar{\mathcal{C}}_{x_j}}}{\mathbb{P}_{p'}(\bar{\mathcal{C}}_{x_j})}\ .$$

Using this we identify the conditional distribution $p_{\vec{U},Y|\vec{X}}$ as:

$$p_{\vec{U},Y|\vec{X}}(\vec{u},y|\vec{x}) := \begin{cases} \bigotimes_{j=1}^{2}\left[\frac{p'_{S_j}(s_j)p'_{M_j|S_j,X_j}(m_j|s_j,x_j)}{\mathbb{P}_{p'}(\bar{\mathcal{C}}_{x_j})}\right]p'_{Y|\vec{S},\vec{M}}(y|\vec{s},\vec{m}), & \text{if } m_j,s_j \in \bar{\mathcal{C}}_{x_j} \\ \left(= \bigotimes_{j=1}^{2}\left[\frac{p'_{U_j|X_j}(u_j|x_j)\mathbb{1}_{m_j,s_j\in\bar{\mathcal{C}}_{x_j}}}{\mathbb{P}_{p'}(\bar{\mathcal{C}}_{x_j})}\right]p'_{Y|\vec{U}}(y|\vec{u})\right), & \\ 0, & otherwise. \end{cases} \tag{109}$$

Now, we identify the complete joint distribution p defined as follows:

$$p_{\vec{X},\vec{U},Y}(\vec{x},\vec{u},y) := \begin{cases} \bigotimes_{j=1}^{2}\left[\frac{q_{X_j}(x_j)p'_{U_j|X_j}(u_j|x_j)}{\mathbb{P}_{p'}(\bar{\mathcal{C}}_{x_j})}\right]p'_{Y|U_1,U_2}(y|u_1,u_2), & \text{if } (\vec{u}) \in \bar{\mathcal{C}}_{\vec{x}} \\ 0, & otherwise. \end{cases} \tag{110}$$

Note that (107) also gives:

$$\max_{\vec{x}}\left\|p_{Y|\vec{X}=\vec{x}} - p'_{Y|\vec{X}=\vec{x}}\right\|_{tvd} = \max_{\vec{x}}\left\|\sum_{\vec{m},\vec{s}}\left(p_{\vec{M},\vec{S}|\vec{X}=\vec{x}} - p'_{\vec{M}\vec{S}|\vec{X}=\vec{x}}\right)p'_{Y|\vec{M}=\vec{m},\vec{S}=\vec{s}}\right\|_{tvd} \tag{111}$$

$$\leq \frac{\max_{\vec{x}}\sum_{(\vec{m},\vec{s})\in\mathcal{C}_{\vec{x}}}|p(\vec{m},\vec{s}|\vec{x}) - p'(\vec{m},\vec{s}|\vec{x})| + \max_{\vec{x}}\sum_{(\vec{m},\vec{s})\in\bar{\mathcal{C}}_{\vec{x}}}|p(\vec{m},\vec{s}|\vec{x}) - p'(\vec{m},\vec{s}|\vec{x})|}{2}$$

$$= \frac{1}{2}\max_{\vec{x}}\mathbb{P}_{p'}(\mathcal{C}_{\vec{x}}) + \frac{1}{2}\max_{\vec{x}}\mathbb{P}_{p'}(\bar{\mathcal{C}}_{x_1})\mathbb{P}_{p'}(\bar{\mathcal{C}}_{x_2})\left(\frac{1}{\mathbb{P}_{p'}(\bar{\mathcal{C}}_{x_1})\mathbb{P}_{p'}(\bar{\mathcal{C}}_{x_2})} - 1\right)$$

$$\leq \varepsilon_1 + \varepsilon_2\ . \tag{112}$$

Finally we define the following distribution on the random variable $U_j(= (M_j,S_j))$ that will be used to evaluate the quantity $I^{\varepsilon}_{\max}(p_{U_j|X_j})$ for $j \in \{1,2\}$:

$$r_{U_j}(u_j) := q_{S_j}(s_j)\frac{1}{|\mathcal{M}_j|} \tag{113}$$

These identifications leads to the following implications on the rate of the protocol:

$$\begin{aligned} 2^{I^{\varepsilon_j}_{\max}(p_{U_j|X_j})} &\overset{(a)}{\leq} 2^{D_{\max}(p'_{X_j,U_j}||p'_{X_j}\times r_{U_j})} \\ &= \max_{x_j}\max_{u_j}\frac{p'_{X_j,U_j}(x_j,u_j)}{p'_{X_j}(x_j)r_{U_j}(u_j)} \\ &\overset{(b)}{=} \max_{x_j}\max_{(m_j,s_j)}\frac{q_{S_j}(s_j)p'_{M_j|S_jX_j}(m_j|s_j,x_j)}{q_{S_j}(s_j)/|\mathcal{M}_j|} \\ &\overset{(c)}{\leq} |\mathcal{M}_j|, \end{aligned} \tag{114}$$

where (a) follows from the definition of channel smoothed $I_{\max}$ in Definition 3 and observing that distribution $p_{U_j|X_j=x_j} = p_{M_j,S_j|X_j=x_j} \in \mathcal{B}^{\varepsilon_j}(p'_{M_j,S_j|X_j=x_j})$ because:

$$\begin{aligned} &\max_{x_j}\left\|p_{U_j|X_j=x_j} - p'_{U_j|X_j=x_j}\right\|_{tvd} \\ &= \max_{x_j}\frac{1}{2}\sum_{m_j,s_j}\left|p'_{M_j,S_j|X_j}(m_j,s_j|x_j) - p_{M_j,S_j|X_j}(m_j,s_j|x_j)\right| \\ &= \max_{x_j}\frac{1}{2}\left[\sum_{m_j,s_j\in\bar{\mathcal{C}}_{x_j}}p'_{M_j,S_j|X_j}(m_j,s_j|x_j)\left(\frac{1}{\mathbb{P}_{p'}(\bar{\mathcal{C}}_{x_j})} - 1\right) + \sum_{m_j,s_j\in\mathcal{C}_{x_j}}p'_{M_j,S_j|X_j}(m_j,s_j|x_j)\right] \\ &= \mathbb{P}_{p'}(\mathcal{C}_{x_j}) \leq \varepsilon_j\ (\text{ from } (107)), \end{aligned}$$

(b) follows from the identification of $U_j = (M_j, S_j)$ for all p' and the Bayes rule and
(c) follows since $p'_{M_j,S_j|X_j}(m_j, s_j|x_j) \leq 1$ and the definition of $r_j(U_j)$.
We thus have from (114), the rate of the code is lower bounded by:

$$R_j = \log |\mathcal{M}_j| \geq I^{\varepsilon_j}_{\max}(p_{U_j|X_j}) \;\text{ for } j \in \{1, 2\}.$$

From (112) we have that $p_{Y|\vec{X}=\vec{x}} \in \mathcal{B}^{\varepsilon_1+\varepsilon_2}(p'_{Y|\vec{X}=\vec{x}})$. This along with the simulation constraint of (104) yields by the triangle inequality:

$$\max_{\vec{x}} \left\| p_{Y|\vec{X}=\vec{x}} - q_{Y|\vec{X}=\vec{x}} \right\|_{tvd} \leq 2(\varepsilon_1 + \varepsilon_2). \tag{115}$$

We have thus identified the auxiliary channels $p_{U_1|X_1}, p_{U_2|X_2}$ and a distribution $p_{Y|U_1,U_2} := p'_{Y|U_1,U_2}$ (from the decoder of the simulation protocol), such that:

$$(p_{U_1|X_1}, p_{U_2|X_2}, p_{Y|U_1,U_2}) \in \mathcal{A}^{outer}_{\varepsilon},$$

where the set $\mathcal{A}^{outer}_{\varepsilon}$ is given in (42) and the rate of any (R_1, R_2, ε)-simulation code is bounded below by:

$$R_j \geq I^{\varepsilon_j}_{\max}(p_{U_j|X_j}).$$

To complete the proof, we require a bound on the cardinalities of $\mathcal{U}_1$, $\mathcal{U}_2$, which is same as given in Lemma 1.3 and proven in Appendix B-B. ■

C. Asymptotic expansion

We will extend the single-letter universal protocol to the asymptotic iid case and show that the rate region can still be single-letterized. We split the proof into two parts showing that the asymptotic inner and outer bounds converge to $\mathcal{R}^{iid}_{\mathbb{U}}$. Throughout the proof, we choose the parameters $\varepsilon_j > 0$ and $\delta \in (0, \min\{\varepsilon_1, \varepsilon_2, 1-\varepsilon_1, 1-\varepsilon_2\})$.

1) **Asymptotic iid Universal Inner Bound:** We extend the universal one-shot inner bound to obtain the optimal universal asymptotic iid rate region. Let $(R_1, R_2) \in \mathcal{R}^{iid}_{\mathbb{U}}$ (defined in (44)) be such that for any $\eta > 0$,

$$R_j \geq \max_{q_{X_j}} I(X_j; U_j)_{q_{X_j} p_{U_j|X_j}} + \eta, \tag{116}$$

for some $p_{U_1,U_2,Y|X} = p_{U|X_1} p_{U_2|X_2} p_{Y|U_1,U_2}$ satisfying $p(y|x_1, x_2) = q(y|x_1, x_2)$, for all x_1, x_2. Consider

$$p_{U_1^n|X_1^n} = p^{\otimes n}_{U_1|X_1}\;, p_{U_2^n|X_2^n} = p^{\otimes n}_{U_2|X_2} \text{ and } p_{Y^n|U_1^n,U_2^n} = p^{\otimes n}_{Y|U_1,U_2}. \tag{117}$$

Note that the triple

$$(p_{U_1^n|X_1^n}, p_{U_2^n|X_2^n}, p_{Y^n|U_1^n,U_2^n}) \in \mathcal{A}^{(n)}_{inner}, \tag{118}$$

where the set $\mathcal{A}^{(n)}_{inner}$ is the n^{th} extension of the set $\mathcal{A}_{inner}$ defined in (39) of Theorem 2. The AEP from Fact 9 for the channel smoothed $\max$-mutual information gives

$$\lim_{n\to\infty} \frac{1}{n} \left[n \cdot \max_{q_{X_j}} I(X_j; U_j)_{q_{X_j} p_{U_j|X_j}} + \log\log\frac{1}{\delta} \right] = \max_{q_{X_j}} I(X_j; U_j)_{q_{X_j} p_{U_j|X_j}}.$$

which from (116) means that

$$nR_j \geq I^{\varepsilon_j-\delta}_{\max}(p^{\otimes n}_{U_j|X_j}) + \log\log\frac{1}{\delta}, \tag{119}$$

for all sufficiently large n (depending on η). Hence, (118) and (119) together, in the asymptotic limit $n \to \infty$ and $\varepsilon_j \to 0$ imply $\mathcal{R}^{iid}_{\mathbb{U}} \subseteq \mathcal{R}^{(n)}_{\mathbb{U},inner}(\varepsilon_1, \varepsilon_2)$, where

$$\mathcal{R}^{(n)}_{\mathbb{U},inner}(\varepsilon_1, \varepsilon_2) = \left\{ (R_1, R_2) : nR_j \geq I^{\varepsilon_j-\delta}_{\max}(p_{U_j^n|X_j^n}) + \log\log\frac{1}{\delta};\; \text{for } j \in \{1, 2\} \right\}. \tag{120}$$

2) **Asymptotic iid Universal Outer Bound:** In order to prove that the asymptotic extension of the universal one-shot outer bound is the region $\mathcal{R}_{\mathbb{U}}^{iid}$, for any $\varepsilon \in (0,1)$ we first define the following ε-approximate universal iid region as follows:

$$\mathcal{R}_{\mathbb{U}}^{iid}(\varepsilon) := \Bigg\{ (R_1, R_2) : R_j \geq \max_{q_{X_j}} I(X_j; U_j)_{q_{X_j} p_{U_j|X_j}}, \; \forall\, p_{X_1,X_2,U_1,U_2,Y} = q_{X_1} q_{X_2} p_{U_1|X_1} p_{U_2|X_2} p_{Y|U_1,U_2}$$
$$\text{such that } \max_{x_1,x_2} \left\| \sum_{u_1,u_2} p(u_1|x_1) p(u_2|x_2) p(y|u_1,u_2) - q(y|x_1,x_2) \right\|_{tvd} \leq \varepsilon \Bigg\}. \tag{121}$$

We will now use the converse of Theorem 2 to an n-letter universal simulation block code with the distribution $p_{U_1^n|X_1^n} p_{U_2^n|X_2^n} p_{Y^n|U_1^n,U_2^n}$. For any $\varepsilon \in (0,1)$ and $\varepsilon_1, \varepsilon_2 > 0$ such that $\max\{\varepsilon_1, \varepsilon_2\} \leq \varepsilon/4$, let $\mathcal{R}_{\mathbb{U},outer}^{(n)}(\varepsilon_1, \varepsilon_2, \varepsilon)$ be the n-fold extension of the region $\mathcal{R}_{\mathbb{U}}^{outer}(\varepsilon_1, \varepsilon_2, \varepsilon)$ with respect to the input and auxiliary random variables $(X_j^n, U_j^n) \sim q_{X_j}^{\otimes n} p_{U_j^n|X_j^n}$. Suppose $(nR_1, nR_2) \in \mathcal{R}_{\mathbb{U},outer}^{(n)}(\varepsilon_1, \varepsilon_2, \varepsilon)$ with $p_{X_1^n,U_1^n,X_2^n,U_2^n,Y^n} := q_{X_1}^{\otimes n} q_{X_2}^{\otimes n} p_{U_1^n|X_1^n} p_{U_2^n|X_2^n} p_{Y^n|U_1^n,U_2^n}$ being the distribution induced by any n-fold simulation code satisfying

$$\max_{x_1^n,x_2^n} \left\| \sum_{u_1^n,u_2^n} p(u_1^n|x_1^n) p(u_2^n|x_2^n) p(y^n|u_1^n,u_2^n) - \bigotimes_{i=1}^{n} q(y_i|x_{1,i},x_{2,i}) \right\|_{tvd} \leq \varepsilon \text{ (as } 2(\varepsilon_1+\varepsilon_2) \leq \varepsilon). \tag{122}$$

Then, we have

$$\begin{aligned}
nR_j &\geq I_{\max}^{\varepsilon_j}(p_{U_j^n|X_j^n}) \\
&\overset{(a)}{=} \max_{q_{X_j^{\otimes n}}} \min_{\bar{p}_{U_j^n|X_j^n} \in \mathcal{B}^{\varepsilon_j}(p_{U_j^n|X_j^n})} \min_{r_{U_j^n}} D_{\max}\left(q_{X_j}^{\otimes n} \bar{p}_{U_j^n|X_j^n} \middle\| q_{X_j}^{\otimes n} \times r_{U_j^n} \right) \\
&\geq \max_{q_{X_j^{\otimes n}}} \min_{\bar{p}_{U_j^n|X_j^n} \in \mathcal{B}^{\varepsilon_j}(p_{U_j^n|X_j^n})} \min_{r_{U_j^n}} D\left(q_{X_j}^{\otimes n} \bar{p}_{U_j^n|X_j^n} \middle\| q_{X_j}^{\otimes n} \times r_{U_j^n} \right) \\
&\overset{(b)}{=} \max_{q_{X_j^{\otimes n}}} \min_{\bar{p}_{U_j^n|X_j^n} \in \mathcal{B}^{\varepsilon_j}(p_{U_j^n|X_j^n})} \min_{r_{U_j^n}} D\left(q_{X_j}^{\otimes n} \bar{p}_{U_j^n|X_j^n} \middle\| q_{X_j}^{\otimes n} \times \left\{ \sum_{x_j^n} q_{X_j}^{\otimes n}(x_j) \bar{p}_{U_j^n|X_j^n = x_j^n} \right\} \right) \\
&= \max_{q_{X_j^{\otimes n}}} \min_{\bar{p} \in \mathcal{B}^{\varepsilon_j}(p_{U_j^n|X_j^n})} I(X_j^n; U_j^n)_{\bar{p}} \\
&\overset{(c)}{\geq} \max_{q_{X_j^{\otimes n}}} \left[I(X_j^n; U_j^n)_{q_{X_j^{\otimes n}} p_{U_j^n|X_j^n}} - 2h_2\left(\frac{\varepsilon_j}{1+\varepsilon_j} \right) - 2n\varepsilon_j \log(|\mathcal{X}_j|) \right] \\
&\overset{(d)}{\geq} \max_{q_{X_j}} \left[nI(X_j; U_j)_{q_{X_j} p_{U_j|X_j}} - 2h_2\left(\frac{\varepsilon_j}{1+\varepsilon_j} \right) - 2n\varepsilon_j \log(|\mathcal{X}_j|) \right]
\end{aligned}$$

where (a) follows since the smoothed channel max-information is independent of the input distribution from (2) (b) follows from the fact that the minimum in $D\left(q_{X_j}^{\otimes n} \bar{p}_{U_j^n|X_j^n} \middle\| q_{X_j}^{\otimes n} \times r_{U_j^n} \right)$ is achieved at $r_{U_j^n} = \sum_{x_j^n} q_{X_j}^{\otimes n}(x_j) \bar{p}_{U_j^n|X_j^n}$; (c) follows from Fact 6 and cardinality bounds on $\mathcal{U}_j$ from Lemma 1.3; and (d) follows by a similar analysis to that of Proposition 1 (by maximizing over inputs $q_{\vec{X}}$) for some $p_{U_1,U_2,Y|X_1,X_2} = p_{U_1|X_1} p_{U_2|X_2} p_{Y|U_1,U_2}$ satisfying $\max_{\vec{x}} \left\| p_{Y|\vec{X}=\vec{x}} - q_{Y|\vec{X}=\vec{x}} \right\|_{tvd} \leq \varepsilon$ due to monotonicity of trace distance.
By taking the limits $\lim_{\varepsilon_j \to 0} \lim_{n \to \infty}$, we get

$$\begin{aligned}
R_j &\geq \lim_{\varepsilon_j \to 0} \lim_{n \to \infty} \max_{q_{X_j}} \left[I(X_j; U_j)_{q_{X_j} p_{U_j|X_j}} - \frac{2h_2\left(\frac{\varepsilon_j}{1+\varepsilon_j} \right)}{n} - 2\varepsilon_j \log(|\mathcal{X}_j|) \right] \\
&= \max_{q_{X_j}} \lim_{\varepsilon_j \to 0} \lim_{n \to \infty} \left[I(X_j; U_j)_{q_{X_j} p_{U_j|X_j}} - \frac{2h_2\left(\frac{\varepsilon_j}{1+\varepsilon_j} \right)}{n} - 2\varepsilon_j \log(|\mathcal{X}_j|) \right]
\end{aligned}$$

$$= \max_{q_{X_j}} I(X_j;U_j)_{q_{X_j} p_{U_j|X_j}}.$$

Hence, we have shown that in the asymptotic iid limit:

$$\lim_{n\to\infty} \mathcal{R}^{(n)}_{\mathbb{U},outer}(\varepsilon_1,\varepsilon_2,\varepsilon) \subseteq \mathcal{R}^{iid}_{\mathbb{U}}(\varepsilon). \tag{123}$$

Since, in our setting we have bounded cardinalities of the auxiliary random variables, we can directly apply [14, Lemma 6] in our case (see Fact 10 for exact statement) to obtain

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty} \mathcal{R}^{(n)}_{\mathbb{U},outer}(\varepsilon) := \mathcal{R}^{iid}_{\mathbb{U},outer} \subseteq \mathcal{R}^{iid}_{\mathbb{U}} = \lim_{\varepsilon\to 0} \mathcal{R}^{iid}_{\mathbb{U}}(\varepsilon).$$

Thus we have shown that in the asymptotic iid limit:

$$\mathcal{R}^{outer}_{\mathbb{U}} \subseteq \mathcal{R}^{iid}_{\mathbb{U}} \subseteq \lim_{n\to\infty} \mathcal{R}^{(n)}_{\mathbb{U},inner} \subseteq \mathcal{R}^{outer}_{\mathbb{U}},$$
$$\Rightarrow \mathcal{R}^{inner}_{\mathbb{U}} := \lim_{\varepsilon_1,\varepsilon_2\to 0}\lim_{n\to\infty} \mathcal{R}^{(n)}_{\mathbb{U},inner}(\varepsilon_1,\varepsilon_2) = \mathcal{R}^{iid}_{\mathbb{U}} = \mathcal{R}^{outer}_{\mathbb{U}}.$$

The following fact states that the region $\lim_{\varepsilon\to 0} \mathcal{R}^{iid}_{\mathbb{U}}(\varepsilon) = \mathcal{R}^{iid}_{\mathbb{U}}$.

Fact 10: Consider the asymptotic iid setting for universally simulating a MAC $q_{Y|X_1,X_2}$ (see Definition 8) for any $\varepsilon \in (0,1)$. Let an outer bound on the cost region for this task be

$$\mathcal{R}^{iid}_{\mathbb{U}}(\varepsilon) := \left\{ (R_1,R_2) : R_j \geq \max_{p_{X_j}} I(X_j;U_j)_{p_{X_j} p_{U_j|X_j}},\ j\in\{1,2\},\ p_{\vec{X},\vec{U},Y}\ s.t.\ \max_{\vec{x}} \left\| p_{Y|\vec{X}=\vec{x}} - q_{Y|\vec{X}=\vec{x}} \right\|_{tvd} \leq \varepsilon \right\},$$

with $|\mathcal{U}_j| \leq |\mathcal{X}_1||\mathcal{X}_2||\mathcal{Y}|$ for $j\in\{1,2\}$ then:

$$\mathcal{R}^{iid}_{\mathbb{U}} = \bigcap_{\varepsilon>0} \mathcal{R}^{iid}_{\mathbb{U}}(\varepsilon),$$

where $\mathcal{R}^{iid}_{\mathbb{U}} := \left\{ R_j \geq \max_{p_{X_j}} I(X_j;U_j)_{p_{X_j} p_{U_j|X_j}},\ \text{for } j\in\{1,2\},\ p_{\vec{X},\vec{U},Y}\ s.t. p_{Y|\vec{X}=\vec{x}} = q_{Y|\vec{X}=\vec{x}} \right\}$.

The proof of the above fact is almost the same as that of Fact 7 using the continuity of the function $\max_{p_{X_j}} I(U_j;X_j)_{p_{X_j} p_{U_j|X_j}}$ with respect to the argument $p_{U_j|X_j}$ (see [27, Lemma 12 and Corollary 13] by considering $p_{U_j|X_j}$ as channels).

Appendix E
One-shot Measurement Compression

A. Task and achievability

We first recall one of the most fundamental theorems of quantum information theory that is used to prove the existence of an isometry which can serve as either encoder or a decoder. This is the following widely used Uhlmann's theorem.

Fact 11: Uhlmann's Theorem [28] Consider (finite dimensional) density matrices ρ^A, σ^A. Let $|\psi_\rho\rangle^{AB}$ be a purification of ρ^A, and let $|\phi_\sigma\rangle^{AC}$ be a purification of σ^A. Then there exists an isometry $V^{C\to B}$ such that,

$$F((I^A\otimes V)\,|\phi_\sigma\rangle\langle\phi_\sigma|\,(I^A\otimes V^\dagger), |\psi_\rho\rangle\langle\psi_\rho|) = F(\rho^A,\sigma^A),\ \text{where } F(\rho,\sigma) := \left\|\sqrt{\rho}\sqrt{\sigma}\right\|_1.$$

We now state a version of Uhlmann's theorem for CQ states, which we shall be using in our achievability proof of Lemma 3.1.

Fact 12: [20, Claim 4] Let $\varphi^{EE'U}, \tau^{EIU}$ be two CQ states of the form

$$\varphi^{EE'U} := \sum_u p(u)\,|u\rangle\langle u|^U \otimes |\varphi_u\rangle\langle\varphi_u|^{EE'} \ \text{and}\ \tau^{EIU} := \sum_u q(u)\,|u\rangle\langle u|^U \otimes |\tau_u\rangle\langle\tau_u|^{EI}.$$

Then there exists a set of isometries indexed by the contents of the classical register U, denoted by $\{V_u^{E'\to I}\}$ such that

$$F\left(\left\{\sum_u |u\rangle\langle u|^U \otimes I^E \otimes V_u\right\}\varphi^{EE'U}\left\{\sum_u |u\rangle\langle u|^U \otimes I^E \otimes V_u^\dagger\right\}, \tau^{EIU}\right) = F\left(\varphi^{EU}, \tau^{EU}\right).$$

We now define the task of one-shot measurement compression with feedback which was studied in [20].

Definition 12: [20, Definition 1] Let $\Lambda^{A\to X}$ be a quantum measurement described as

$$\eta^{EX} := \Lambda^{A\to X}(\psi^{EA}) := \sum_x \mathrm{Tr}[\Lambda_x \rho^A]\, |x\rangle\langle x|^X \otimes \psi_x^E$$

and let $|\psi\rangle^{EA}$ be any purification of ρ^A. Then for any given $\varepsilon \in (0,1)$, a purification $|\eta\rangle^{EE'XX'}$ of η^{EX}, with $\mathcal{X} \cong \mathcal{X}'$, an (R,ε)-quantum measurement compression protocol with feedback consists of:

- A pre-shared random state $S^{A'A''}$ between the sender and the receiver
- A rate limited noiseless classical channel of rate R;
- Encoder $\mathcal{E}_{\text{meas. comp.}} : \mathcal{H}^{X'} \otimes \mathcal{H}^S \to [1:2^R]$ and a Decoder $\mathcal{D}_{\text{meas. comp.}} : [1:2^R] \otimes \mathcal{H}^S \to \mathcal{H}^X$ such that

$$\left\|\mathcal{D}_{\text{meas. comp.}} \circ \mathcal{E}_{\text{meas. comp.}}(\eta^{EE'XX'} \otimes S^{A'A''}) - \eta^{EE'XX'}\right\|_{tvd} \leq \varepsilon.$$

Note that the amount of classical communication required for the above task is R bits.

An achievable rate for the above measurement compression task was given in [20, Theorem 1]. To obtain a one-shot achievable rate for simulation of CS-QC MAC with feedback, we will employ the protocol of [20] individually for each sender. However, we characterize the required rate in terms of our definition of $I^{\varepsilon}_{\max}$ (see (3)) by modifying the analysis slightly. The exact statement of the convex split lemma for CQ states with its characterization in terms of $I^{\varepsilon}_{\max}$ according to (3) is given in Fact 3.

We further recall that the task of quantum measurement compression with feedback is very similar to that of quantum state splitting [21]. Both these tasks, in turn use convex split lemma of Fact 3 to quantify the rate. We also use Fact 3 to derive our CS-QC MAC with feedback simulation rate region and hence the minute distinction between measurement compression with feedback and quantum state splitting is not useful for our purpose. In fact we essentially prove the achievability for the task of Definition 12 (similar to that of [20, Theorem 1], but with a slightly different definition of $I^{\varepsilon}_{\max}$) to derive $\mathcal{R}^{QC-fb}_{inner}(\varepsilon_1, \varepsilon_2, \delta)$ given by (51) in Definition 11. This is because we can equivalently see our encoding as the simulation of the post-measurement states $|\varphi_j\rangle^{E_jE'_jX_jX''_jU_j\tilde{U}_j}$ by compressing $\tilde{U}_j$ and allowing the receiver to reconstruct it, so that the overall joint-state shared with the receiver is still close to $|\varphi_j\rangle^{E_jE'_jX_jX''_jU_j\tilde{U}_j}$ and receiver holds the register $\tilde{U}_j$ relabelled as $\bar{U}_j$ in Lemma 3.1 (see (58)).

B. Proof of Lemma 3.1

We give a self-contained proof of the achievable rates mentioned in Lemma 3.1. This follows by the direct application of the convex split lemma to the states $|\varphi_j\rangle^{E_jE'_jX_jX''_jU_j\tilde{U}_j}$ with the shared randomness as n_j-fold iid copy of $S_j^{U'_jU''_j} := \sum_{u_j} \tilde{p}_{U_j}(u_j)\, |u_j\rangle\langle u_j|^{U'_j} \otimes |u_j\rangle\langle u_j|^{U''_j}$, where the registers $U'_{j,1}, \ldots, U'_{j,n_j}$ are held by the sender and $U''_{j,1}, \ldots, U''_{j,n_j}$ are held by the receiver. Further, the distribution $\tilde{p}_{U_j}$ of the shared randomness is the optimizing distribution on U_j in the definition $I^{\varepsilon_j-\delta}_{\max}(E_j;U_j)_{\varphi_j}$ (see (3)). Recall that the CQ state $|\varphi_j\rangle$ from (58) and its reduced state $\varphi_j^{E_j\tilde{U}_j}$ from (57) are given as:

$$|\varphi_j\rangle^{E_jE'_jX_jX''_jU_j\tilde{U}_j} = \sum_{x_j,u_j} \sqrt{p_{X_j,U_j}(x_j,u_j)}\, |x_jx_j\rangle^{X_jX''_j}\, |u_ju_j\rangle^{U_j\tilde{U}_j}\, |\varphi_{x_j}\rangle^{E_jE'_j} \text{ and}$$

$$\varphi_j^{E_j\tilde{U}_j} = \sum_{u_j} p_{U_j}(u_j)\, |u_j\rangle\langle u_j|^{\tilde{U}_j} \otimes \tilde{\varphi}^{E_j}_{u_j}, \text{ with } \tilde{\varphi}^{E_j}_{u_j} = \sum_{x_j} p_{X_j|U_j}(x_j|u_j)\varphi^{E_j}_{x_j}.$$

We will thus use the convex split lemma from Fact 3 for transmitting $\tilde{U}_j$ to the receiver such that the correlation of U_j with the (untouched) environment E_j is (almost) preserved. We define the following convex split state and its CQ extension with the quantum system of the CQ state being pure, for each sender j:

$$\mu_j^{E_j U'_{j,1},\dots,U'_{j,n_j}} := \frac{1}{n_j}\sum_{i_j=1}^{n_j} \varphi_j^{E_j U'_{j,i_j}} \bigotimes_{k\neq i_j} S_j^{U'_{j,k}},$$

$$\mu_j^{E_j I_j X_j X''_j U'_{j,1} U''_{j,1} \dots U'_{j,n_j} U''_{j,n_j}} := \sum_{u_1,\dots,u_{n_j}} \tilde{p}_{U_1,\dots,U_{n_j}}(u_1,\dots,u_{n_j}) \left|u_1\dots u_{n_j}\right\rangle\!\left\langle u_1\dots u_{n_j}\right|^{U'_{j,1}\dots U_{j,n'_j}}$$

$$\otimes \left|u_1\dots u_{n_j}\right\rangle\!\left\langle u_1\dots u_{n_j}\right|^{U''_{j,1}\dots U''_{j,n_j}} \otimes \left\{ \left(\sum_{i_j=1}^{n_j} \frac{1}{\sqrt{n_j}} \left|i_j\right\rangle^{I_j} \left|\varphi_{u_{i_j}}\right\rangle^{E_j E'_j X_j X''_j}\right) \left(\sum_{i_j=1}^{n_j} \frac{1}{\sqrt{n_j}} \left\langle i_j\right|^{I_j} \left\langle\varphi_{u_{i_j}}\right|^{E_j E'_j X_j X''_j}\right) \right\}, \tag{124}$$

where $|\varphi_{u_{i_j}}\rangle^{E_j E'_j X_j X''_j} = \sum_{x_j} \sqrt{p_{X_j|U_j}(x_j|u_{i_j})}\, |\phi_{x_j}\rangle^{E_j E'_j} |x_j x_j\rangle^{X_j X''_j}$. Similarly, we have the following CQ extension with the quantum system being pure, of the state $\varphi_j^{E_j} \otimes \left(\bigotimes_{i_j=1}^{n_j} S_j^{U'_{j,i_j}}\right)$:

$$\varphi_j^{E_j E'_j X_j X''_j U_j \tilde{U}_j U'_{j,1} U''_{j,1} \dots U'_{j,n_j} U''_{j,n_j}} = \sum_{u_1,\dots,u_{n_j}} \tilde{p}_{U_{j,1},\dots,U_{j,n_j}}(u_1,\dots,u_{n_j}) \left|u_1,\dots,u_{n_j}\right\rangle\!\left\langle u_1,\dots,u_{n_j}\right|^{U'_{j,1}\dots U'_{j,n_j}}$$

$$\otimes \left|u_1,\dots,u_{n_j}\right\rangle\!\left\langle u_1,\dots,u_{n_j}\right|^{U''_{j,1}\dots U''_{j,n_j}} \otimes |\varphi_j\rangle\!\langle\varphi_j|^{E_j E'_j X_j X''_j U_j \tilde{U}_j}. \tag{125}$$

Convex split lemma from Fact 3 implies that for $\log n_j \geq I_{\max}^{\varepsilon_j-\delta}(E_j;U_j) + 2\log\frac{1}{\delta}$, it holds that:

$$\left\| \varphi_j^{E_j} \bigotimes_{k=1}^{n_j} S_j^{U'_{j,k}} - \mu_j^{E_j U'_{j,1},\dots,U'_{j,n_j}} \right\|_{tvd} \leq \varepsilon_j\,. \tag{126}$$

We can now apply the CQ Uhlmann's theorem from Fact 12 due to the desired structure of the CQ extensions of μ_j and φ_{u_j} in equations (124) and (125) , respectively. Thus, from Fact 12 there exists a conditional isometry $V_{u_1,\dots,u_{n_j}}^{E'_j X_j X''_j U_j \tilde{U}_j \to E'_j X_j X''_j I_j}$ such that, for

$$\tilde{\nu}_j^{E_j I_j U'_{j,1} U''_{j,1} \dots U'_{j,n_j} U''_{j,n_j}} := \left(\sum_{u_1,\dots,u_{n_j}} \left|u_1\dots u_{n_j}\right\rangle\!\left\langle u_1\dots u_{n_j}\right|^{U'_{j,1}\dots U'_{j,n_j}} \otimes V_{u_1,\dots,u_{n_j}}\right) |\varphi_j\rangle\!\langle\varphi_j| \otimes S_j$$

$$= \sum_{u_1,\dots,u_{n_j}} \tilde{p}_{U_{j,1},\dots,U_{j,n_j}}(u_1,\dots,u_{n_j}) \left|u_1\dots u_{n_j}\right\rangle\!\left\langle u_1\dots u_{n_j}\right|^{U'_{j,1}\dots U'_{j,n_j}} \otimes \left|u_1\dots u_{n_j}\right\rangle\!\left\langle u_1\dots u_{n_j}\right|^{U''_{j,1}\dots U''_{j,n_j}} \tag{127}$$

$$\otimes |\tilde{\varphi}_j\rangle\!\langle\tilde{\varphi}_j|^{E_j X_j X''_j I_j}\,, \tag{128}$$

where

$$|\tilde{\varphi}_j\rangle^{E_j E'_j X_j X''_j I_j} := \sum_{x_j} \sqrt{\tilde{p}_{X_j|U_j}(x_j|u_{i_j})}\, |\varphi_{x_j}\rangle^{E_j E'_j} |x_j\rangle^{X_j} \otimes |x_j\rangle^{X''_j} \otimes |i_j\rangle^{I_j}\,,$$

i_j depends on the contents of shared randomness $u_1,\dots,u_{n_j}$ and the classical registers $U_j, \tilde{U}_j$ (since U_j is classically correlated with E_j via X_j, the action of the isometry above holds without any loss of generality), we get:

$$\left\| \tilde{\nu}_j^{E_j E'_j X_j X''_j I_j U'_{j,1} U''_{j,1} \dots U'_{j,n} U''_{j,n_j}} - \mu_j^{E_j E'_j X_j X''_j I_j U'_{j,1} U''_{j,1} \dots U'_{j,n_j} U''_{j,n_j}} \right\|_{tvd} \leq \varepsilon_j. \tag{129}$$

• We now give a protocol that allows the receiver to recover the state $\tilde{\varphi}_j^{E_j X_j \bar{U}_j} \overset{\varepsilon_j}{\sim} \varphi_j^{E_j X_j \tilde{U}_j}$ (by recovering $\bar{U}_j$ from $\tilde{U}_j$):

1) $\mathcal{E}_j$ has the state $|\varphi_j\rangle\!\langle\varphi_j|^{E_j E'_j X_j X''_j U_j \tilde{U}_j}$ as input with access to the random state $\bigotimes_{k=1}^{n_j} S_j^{U'_{j,k} U''_{j,k}}$ shared with the receiver. $\mathcal{E}_j$ applies the conditional isometry $V_{u_1,\dots,u_{n_j}}$, conditioned on the contents of the shared randomness

register from (127), to the input and obtains the state $\tilde{\nu}_j$. This creates the necessary correlation between E_j and the shared random state S_j and is recorded in the register I_j.

2) $\mathcal{E}_j$ then measures the register I_j and sends the classical message i_j using $\log n_j$ bits to the receiver. Thus, the rate of the protocol is (from Fact 3):

$$R_j := \log n_j \geq I_{\max}^{\varepsilon_j - \delta}(E_j; U_j)_\tau + 2\left(\log \frac{1}{\delta}\right) \ . \tag{130}$$

Note that $\log n_j$ here is the rate R_j of the main CS-QC MAC simulation protocol for Lemma 3.1.

3) The final overall state is $\tilde{\varphi}_j^{E_j X_j \bar{U}_j}$. Using (129) the encoder $\mathcal{E}_j$ can pretend as if step 2 is applied on the state μ_j as input, which would have resulted in the overall output state of the protocol with $U_j \sim p_{U_j}$ at the receiver's end. Thus, the encoder for our CS-QC MAC simulation protocol is

$$\mathcal{E}_{j,meas.comp} = \{|i_j\rangle\langle i_j|\}^{I_j} \circ \sum_{u_1,\ldots,u_{n_j}} |u_1 \ldots u_{n_j}\rangle\langle u_1 \ldots u_{n_j}|^{U'_{j,1} \ldots U'_{j,n_j}} \otimes V_{u_1 \ldots u_{n_j}}^{E'_j X_j X''_j U_j \tilde{U}_j \to E'_j X_j X''_j I_j},$$

where $\{|i_j\rangle\langle i_j|\}^{I_j}$ denotes the measurement in the computational basis $|i_j\rangle\langle i_j|$ for the j^{th}-sender. The receiver picks up the shared random register U''_{j,i_j} given in (127) and relabels it to $\bar{U}_j$, as its finally recovered state.

4) Let the step 2 of the encoder $\mathcal{E}_j$ measuring I_j and transmitting the measurement outcome i_j and the step 3 of the receiver recovering $\bar{U}_j$ from U''_{j,i_j} register be represented as a quantum operation $\mathcal{O}_j$.
Thus we have:

$$\left\| \varphi_j^{E_j X_j U_j} - \tilde{\varphi}_j^{E_j X_j \bar{U}_j} \right\|_{tvd} \overset{(a)}{=} \left\| \mu_j^{E_j X_j U_j} - \mathcal{O}_j \left(|\varphi_j\rangle\langle \varphi_j| \bigotimes_{k=1}^{n_j} |S_j\rangle\langle S_j| \right) \right\|_{tvd} \overset{(b)}{\leq} \varepsilon_j \ , \tag{131}$$

where (a) follows from steps 2 and 3 of the protocol above defining $\mathcal{O}_j$; (b) follows from the equation (129) and the monotonicity of the total variation distance. Hence, from equation (131) we get that:

$$\left\| p_{X_j,U_j} - \tilde{p}_{X_j,U_j} \right\|_{tvd} \leq \varepsilon_j. \tag{132}$$

To finish the protocol, the decoder of the CS-QC MAC simulation achievability protocol use these states $\tilde{\varphi}_j^{\bar{U}_j}$ to generate $Y \sim p_{Y|\bar{U}_1,\bar{U}_2}$. Note that the quantum operation $\mathcal{O}_j = \mathcal{D}_{j,\text{meas.comp.}} \circ \mathcal{E}_{j,\text{meas.comp.}}$ is the encoder-decoder operation for each sender, before the final decoding of U_1, U_2 to obtain the desired output Y.

Remark 3.3: We note that the encoding above is essentially the same as that of [20, Theorem 1]. The only difference is that the aforementioned reference proves the achievability for one-shot measurement compression with feedback using a different definition of the smoothed max-mutual information than our Definition 4. If we employ the achievability of [20, Theorem 1] with the definition of $I_{\max}^{\varepsilon}$ considered therein, we get different slack factors. A direct comparison of the different definitions of smoothed max-mutual information (including our Definition 4) can be found in [29].