Technische Universität Braunschweig

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Communication with Unreliable Entanglement Assistance

Quantum Breakfast

Jonas Hawellek, June 26, 2024

Motivation

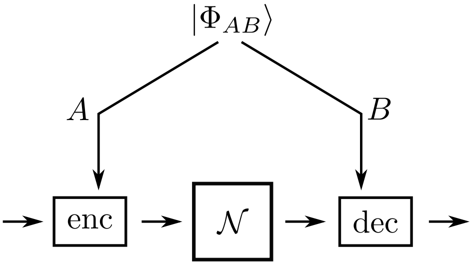
- classical communication over quantum channels
- shared entanglement can help increase communication rates
  - not always available

  $\Rightarrow$ entanglement-assisted strategies become unreliable
- consider a potential loss of entanglement in the coding strategy

Contents

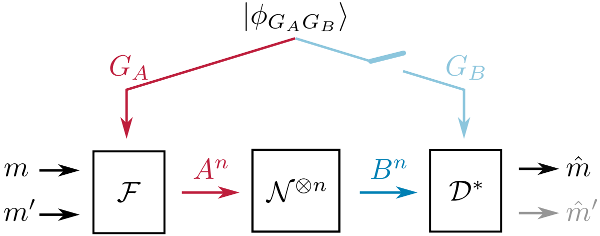
Super-Dense Coding

- pre-shared entangled pair $|\Phi_{AB}\rangle$
- noiseless quantum channel $\mathcal{N}$
- encoding on the system A
- decoding by measuring both particles
- classical capacity doubled[1]
  - 1 bit $\rightarrow$ 2 bit



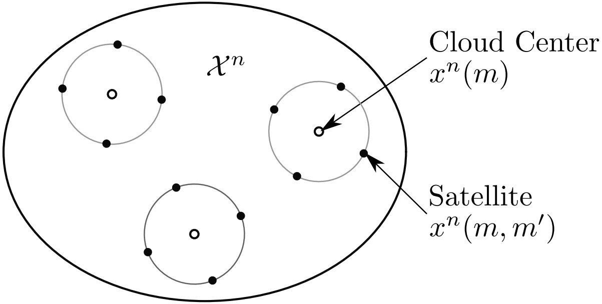
[1] M. M. Wilde. *Quantum Information Theory*. 2nd edition. Cambridge: Cambridge University Press, 2017, Chapter 6.2.3.

Unreliable Entanglement Assistance



- $\mathcal{F}$ encodes message onto G_A
- G_B is available by chance
- decoder $\mathcal{D}$ depends on availability of G_B

Coding and Rates



- encoder $\mathcal{F}$ takes two steps
  - encoding of m
  - superposition-coding of m'
- decoder $\mathcal{D}$ always decodes m
  - guaranteed rate R
- when possible, decode m'
  - excess rate R'

Classical Capacity

Let $\mathcal{N}_{A\to B}$ be a given channel, and define

$$\mathcal{R}_{EA^*}(\mathcal{N}) = \bigcup_{p_X,\phi_{G_A G_B},\mathcal{F}^{(x)}} \left\{ (R, R') : \begin{array}{l} R \leq I(X;B)_\omega \\ R' \leq I(G_B;B|X)_\omega \end{array} \right\}$$

Theorem

The classical capacity region of a quantum channel $\mathcal{N}_{A\to B}$ with unreliable entanglement assistance satisfies [2]

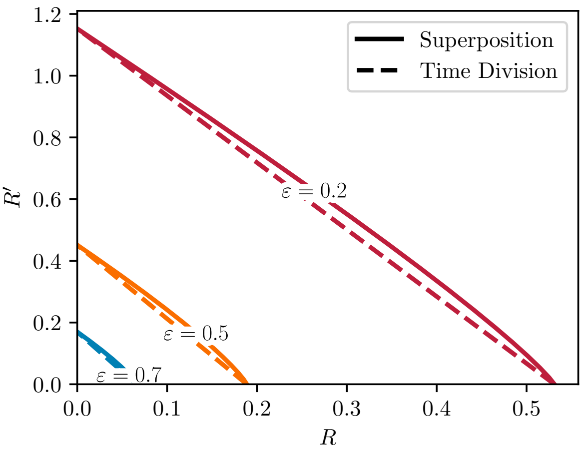
$$\mathcal{C}_{EA^*}(\mathcal{N}) = \bigcup_{n=1}^{\infty} \frac{1}{n}\mathcal{R}_{EA^*}(\mathcal{N}^{\otimes n}).$$

[2] U. Pereg, C. Deppe, and H. Boche. “Communication with unreliable entanglement assistance”. In: *IEEE Trans. Inf. Theory* 69.7 (July 2023), pp. 4579–4599.

Coding Strategies

- time division of entanglement assistance
  - guaranteed rate R for some time
  - unreliable excess rate R' for the rest of time
- superposition strategy
  - $|\varphi\rangle = \sqrt{\beta}|\Phi\rangle + \sqrt{1-\beta}|00\rangle$
  - exploit entanglement to some extent

Depolarizing Channel

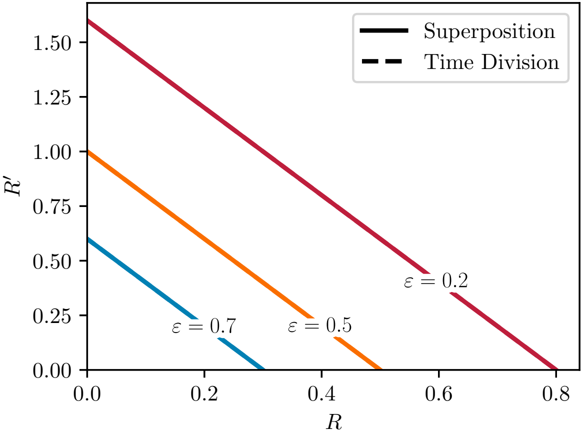


$$\rho \to (1-\varepsilon)\rho + \varepsilon\frac{I}{2}$$

- advantage from superposition
  - optimal for[3] $\varepsilon \geq \frac{2}{3}$

[3] U. Pereg. "Communication over entanglement-breaking channels with unreliable entanglement assistance". In: *Phys. Rev. A* 108.4 (Oct. 2023), p. 042616.

Erasure Channel I



$$\rho \to (1-\varepsilon)\rho + \varepsilon|e\rangle\langle e|$$

- no advantage from superposition
- time division is optimal for independent channel input states

Erasure Channel II

Theorem

The rate region of a qubit erasure channel $\mathcal{N}$ with unreliable entanglement assistance and the input states being constrained to be independent of each other is given by

$$\mathcal{R}_{EA^*}(\mathcal{N}) = \bigcup_{0\leq\lambda\leq 1} \left\{ \begin{array}{ll} (R, R') : & R \leq (1-\lambda)(1-\varepsilon) \\ & R' \leq \lambda(2(1-\varepsilon)) \end{array} \right\} , \quad (1)$$

where $\lambda \in [0, 1]$.

- not valid for arbitrary input states
- are these expressions in general additive?

Stochastic Model for the Availability of Entanglement Assistance I

Motivation

- model of unreliable entanglement assistance allows for worst-/best-case analysis
- physical effects for loss of entanglement are known
- effects can be quantified
- provide an outage probability to sender and receiver
- optimize coding strategy for a mean communication rate

Stochastic Model for the Availability of Entanglement Assistance II

- introduce an entanglement outage probability p_{out}
- define the mean communication rate using $(R, R') \in \mathcal{R}_{EA^*}(\mathcal{N})$

$$R_{\text{mean}} = R + (1 - p_{\text{out}})R'$$

Differentiation from Other Approaches

Limited Entanglement Assistance [4]

- entangled pairs are in a pure state
- entanglement assistance reliably available
- availability limited to some (known) channel uses

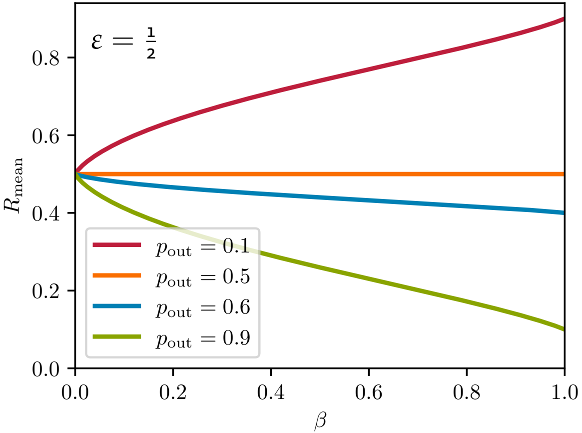
Noisy Entanglement Assistance [5]

- entangled pairs are in a mixed state
- entanglement assistance reliably available
- noise level limits the maximum exploitable amount of entanglement

[4] P. W. Shor. *The Classical Capacity Achievable by a Quantum Channel Assisted by Limited Entanglement.* Feb. 2004. arXiv: quant-ph/0402129.

[5] Q. Zhuang, E. Y. Zhu, and P. W. Shor. "Additive classical capacity of quantum channels assisted by noisy entanglement". In: *Phys. Rev. Lett.* 118.20 (May 2017), p. 200503.

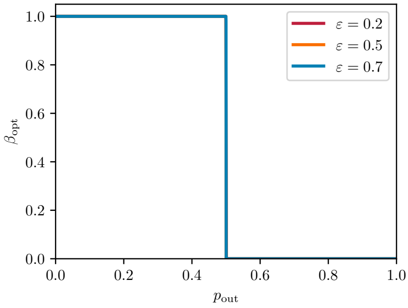
Erasure Channel I

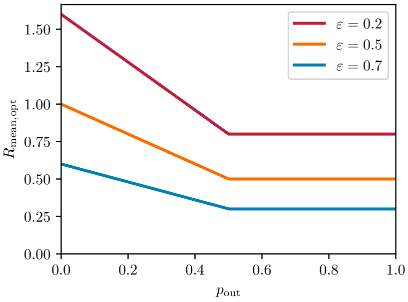


$$\rho \to (1-\varepsilon)\rho + \varepsilon|e\rangle\langle e|$$

- p_{out} is low:
  - rely on entanglement assistance
- p_{out} is high:
  - ignore entanglement assistance

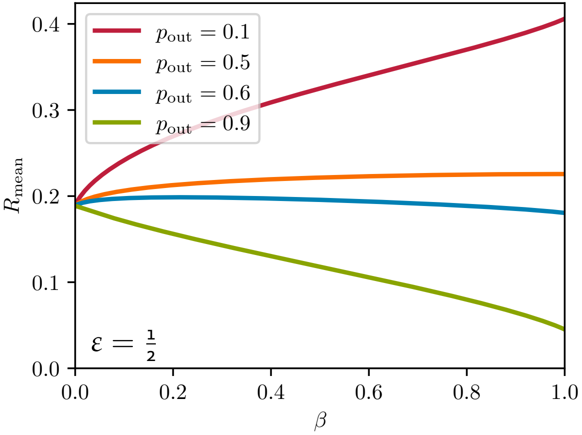
Erasure Channel II





- change strategy at $p_{\text{out}} = \frac{1}{2}$
- rate decreases until unassisted strategy is superior

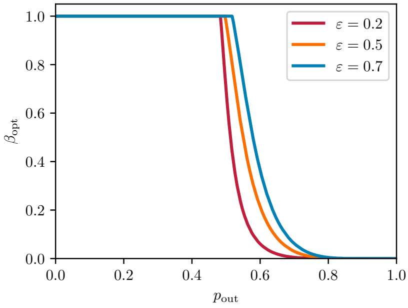
Depolarizing Channel I

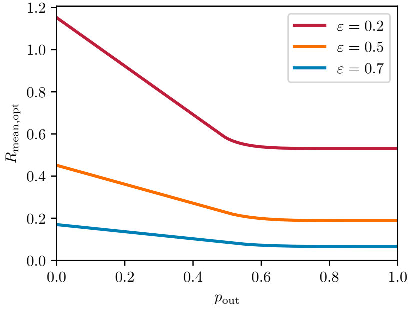


$$\rho \to (1 - \varepsilon)\rho + \varepsilon \frac{I}{2}$$

- p_{out} is low:
  - rely on entanglement assistance
- p_{out} increases:
  - superposition strategy

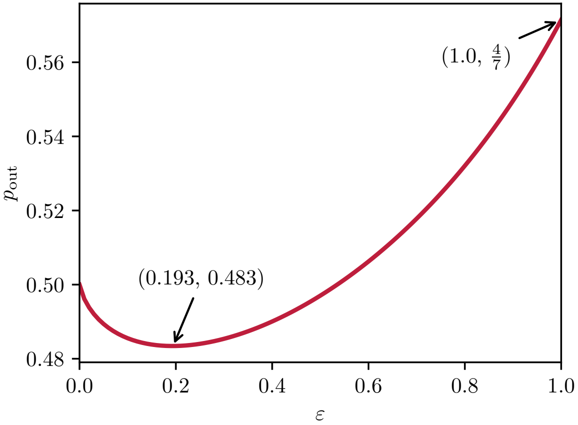
Depolarizing Channel II





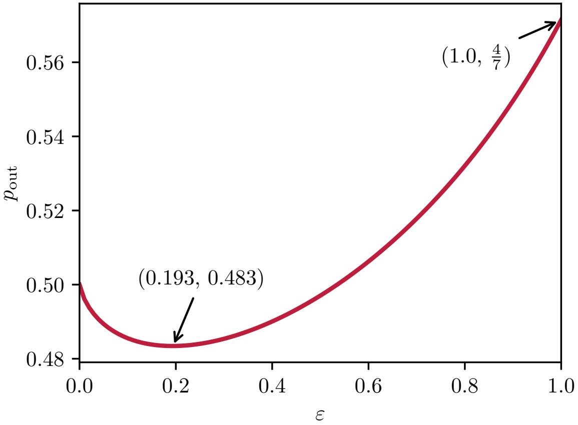
- the region where the optimal strategy changes depends on ε

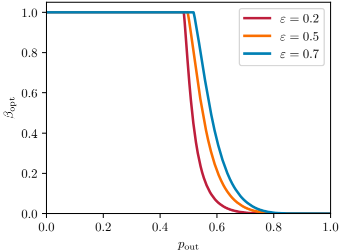
Depolarizing Channel III



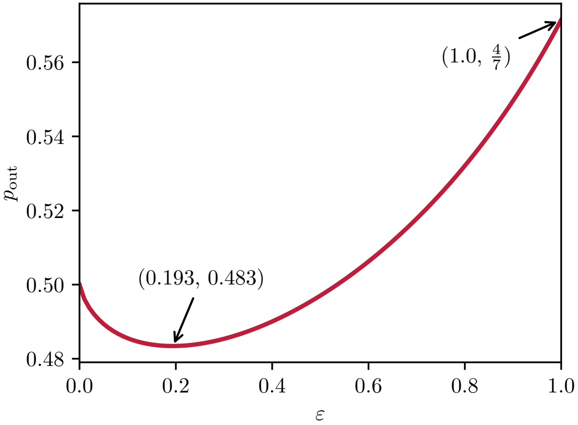
- value for p_{out} where fully entanglement-assisted encoding stops to be optimal
- first decreases to a minimum
- later increases continuously
- extreme values are unexpected

Depolarizing Channel III





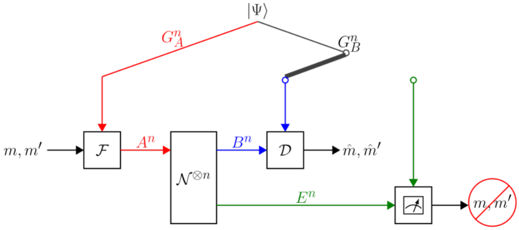
Depolarizing Channel III



- value for p_{out} where fully entanglement-assisted encoding stops to be optimal
- first decreases to a minimum
- later increases continuously
- extreme values are unexpected

Secure Communication with Unreliable Entanglement

- wiretap channel $\mathcal{N}_{A \to BE}$
  - adversary has access to environment E
- entanglement unreliable, since
  - photon gets lost
  - adversary may intercept G_B



(figure cf. [6])

[6] M. Lederman and U. Pereg. *Secure Communication with Unreliable Entanglement Assistance.* Jan. 2024. arXiv: 2401.12861.

Secrecy Capacity

Let $\mathcal{N}_{A\to BE}$ be a given channel, and define

$$\mathcal{R}_{S-EA^*}(\mathcal{N}) = \bigcup_{p_X,\phi_{G_A G_B},\mathcal{F}^{(x)}} \left\{ (R,R') : \begin{array}{l} R \leq [I(X;B)_\omega - I(X;EG_B)_\omega]_+ \\ R' \leq [I(G_B;B|X)_\omega - I(G_B;E|X)_\omega]_+ \end{array} \right\}$$

Theorem

The secrecy capacity region of a degraded quantum wiretap channel $\mathcal{N}_{A\to BE}$ with unreliable entanglement assistance satisfies [7]

$$\mathcal{C}_{S-EA^*}(\mathcal{N}) = \bigcup_{n=1}^{\infty} \frac{1}{n}\mathcal{R}_{S-EA^*}(\mathcal{N}^{\otimes n}).$$

[7] M. Lederman and U. Pereg. *Secure Communication with Unreliable Entanglement Assistance.* Jan. 2024. arXiv: 2401.12861.

Summary & Outlook

Summary

- model of unreliable entanglement assistance
  - classical capacity
  - application to channel models
- extension by a stochastic model
  - application to channel models
- secure communication

Outlook

- further analyze behavior of the depolarizing channel
- consider further channel models
  - implementable ones
- apply model to MAC and BC

Thank You for Your Attention!

Questions?

Comments?

Capacity Theorems

Theorem (Holevo-Schumacher-Westmoreland[8,9])

The classical capacity of a quantum channel $\mathcal{N}_{A\to B}$ is given by

$$\mathcal{C}(\mathcal{N}) = \lim_{k\to\infty} \frac{1}{k}\left(\max_{\rho_{XB}} I(X;B)_\rho\right) , \tag{2}$$

where

$$\rho_{XB} = \sum_{x\in\mathcal{X}} p_X(x)|x\rangle\langle x|_X \otimes \mathcal{N}_{A\to B}^{\otimes k}(\varphi_x) , \tag{3}$$

and φ_x are states for system A at the channel inputs associated with some classical symbol $x \in \mathcal{X}$.

8 A. S. Holevo. “The capacity of the quantum channel with general signal states”. In: *IEEE Trans. Inf. Theory* 44.1 (Jan. 1998), pp. 269–273.

9 B. Schumacher and M. D. Westmoreland. “Sending classical information via noisy quantum channels”. In: *Phys. Rev. A* 56.1 (July 1997), pp. 131–138.

Capacity Theorems

Theorem (Bennett-Shor-Smolin-Thapliyal[10,11])

The classical capacity of a quantum channel $\mathcal{N}_{A\to B}$ with reliable entanglement assistance is given by

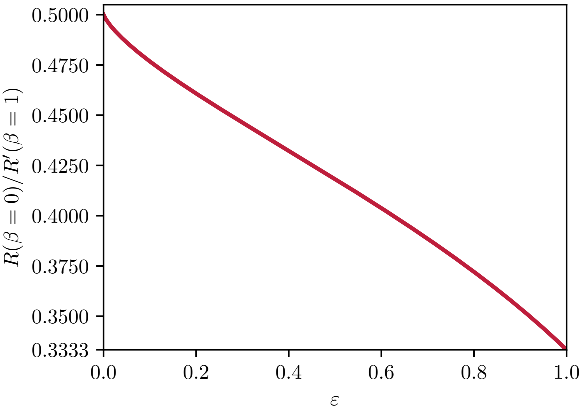
$$C_{EA} = \max_{\phi_{AG_B}} I(B; G_B)_\omega \ , \tag{4}$$

where ϕ_{AG_B} is a pure bipartite state and $\omega_{BG_B} = \mathcal{N}_{A\to B}(\phi_{AG_B})$.

[10] C. H. Bennett et al. "Entanglement-assisted classical capacity of noisy quantum channels". In: *Phys. Rev. Lett.* 83.15 (Oct. 1999), pp. 3081–3084.

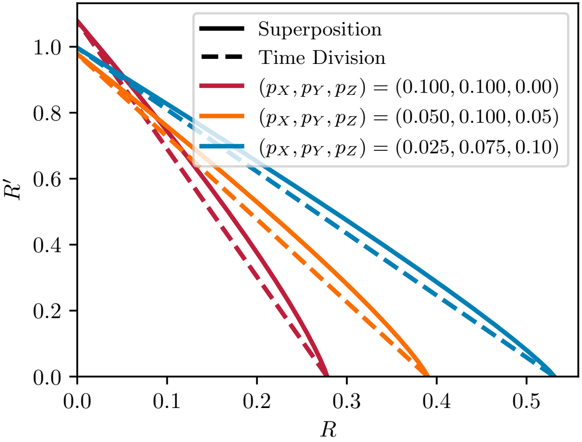
[11] C. H. Bennett et al. "Entanglement-assisted capacity of a quantum channel and the reverse Shannon theorem". In: *IEEE Trans. Inf. Theory* 48.10 (Oct. 2002), pp. 2637–2655.

Depolarizing Channel



- comparing fully entanglement-assisted encoding with the unassisted strategy
- relative relevance changes
- we expect the unassisted strategy to get less relevant with increasing ε (not p_{out})

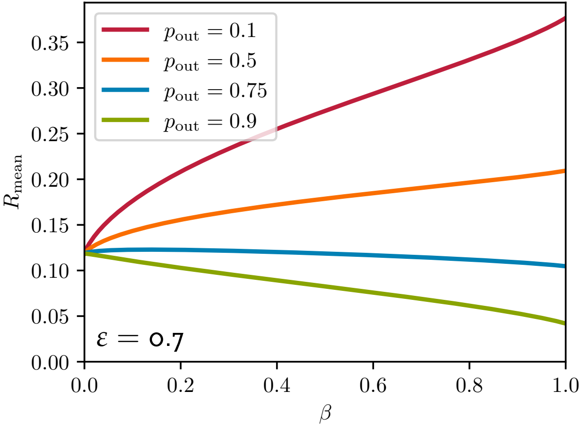
Pauli Channel



$$\rho \to p_I \rho + p_X X \rho X + p_Y Y \rho Y + p_Z Z \rho Z$$

- shape of rate regions changes for different parameter combinations
- superposition outperforms time division in general

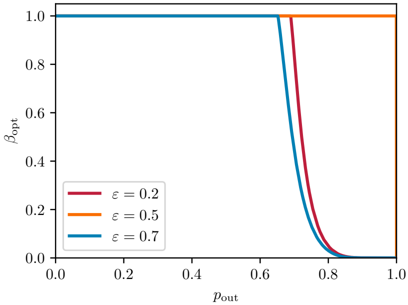
Pauli Channel (Special Case)

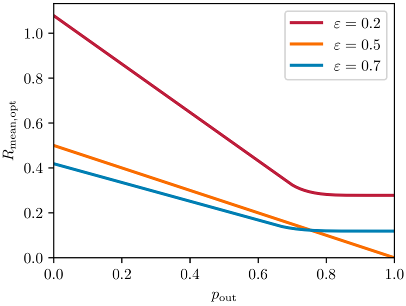


$$\rho \to (1-\varepsilon)\rho + \frac{\varepsilon}{2} X\rho X + \frac{\varepsilon}{2} Y\rho Y$$

- p_{out} is low:
  - rely on entanglement assistance
- p_{out} increases:
  - superposition strategy

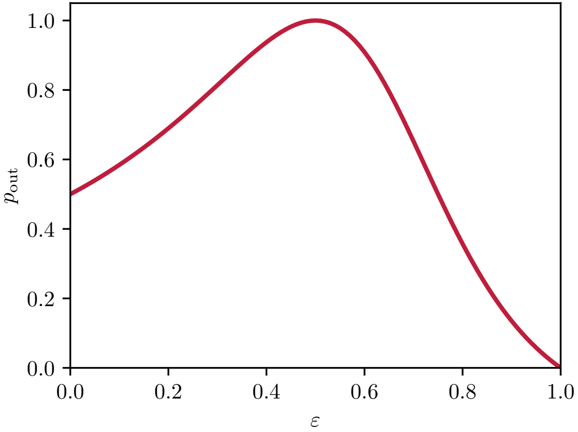
Pauli Channel (Special Case)





- curves are highly dependent on ε
- for $\varepsilon = \frac{1}{2}$ there is no guaranteed rate

Pauli Channel (Special Case)



$$\rho \to (1-\varepsilon)\rho + \frac{\varepsilon}{2} X\rho X + \frac{\varepsilon}{2} Y\rho Y$$

- value for p_{out} where fully entanglement-assisted encoding stops to be optimal
- $\varepsilon = 0$: noiseless channel
- $\varepsilon = \frac{1}{2}$: bit flips randomize output
- $\varepsilon = 1$: phase flips randomize output