Theoretical Computer Science 1
Exercise Sheet 7

René Maseli
Prof. Dr. Roland Meyer
TU Braunschweig
Winter Semester 2024/25

Release: 2024-12-10 Due: 2024-12-17 08:00

Solve the exercises on this sheet and mark on Stud.IP **until Tuesday**, 12/17/2024 **08:00 AM**, which solutions you are capable of presenting in the tutorials. You get the sum of points for those exercises, when you successfully presented a solution, (or if none was required of you).

Homework Exercise 7.1: Costs of Determinization 1 [3 points]

Here we want to show that some languages that admit a description by small NFAs do not admit a description by small DFAs; every DFA for that language is necessarily large.

Given $\Sigma = \{a, b\}$. Consider for all numbers $k \in \mathbb{N}, k > 0$ the language $L_k = \Sigma^*.a.\Sigma^{k-1}$ of words, that have an a at the k-th last position.

a) Show how to construct for any $k \in \mathbb{N}, k > 0$ an NFA $A_k = \langle Q_k, q_0, \rightarrow, F_k\rangle$ with $\mathcal{L}(A_k) = L_k$ and $|Q_k| = k + 1$. Give the automaton formally as a tuple.
   You do not have to show correctness of your construction.

b) Now draw A_3 and its determinization $\mathcal{P}(A_3)$ via Rabin-Scott-power set construction.

Homework Exercise 7.2: Costs of Determinization 2 [2 points]

Consider the languages L_k from exercise 7.1.

Show for all $k \in \mathbb{N}, k > 0$ and $u, v \in \Sigma^k$ with $u \neq v$ the proposition $u \not\equiv_{L_k} v$.

What can you derive for the size of all DFAs for L_k?

Homework Exercise 7.3: Theorem of Myhill & Nerode [4 points]

Let $L \subseteq \Sigma^*$ be a regular language with $\text{Index}(\equiv_L) = k \in \mathbb{N}$ and let $A = \langle Q, q_0, \rightarrow, Q_F\rangle$ be a DFA with $L = \mathcal{L}(A)$ and $|Q| = k$. Let further $A_L = \langle Q_L, q_{0L}, \rightarrow_L, Q_{FL}\rangle$ be the equivalence automaton for L with $\mathcal{L}(A_L) = L$ and $u_1, \dots, u_k$ be the representants of the equivalence classes of $\equiv_L$.

Show Theorem 6.11 from the script: A and A_L are isomorphic. The isomorphism $\beta : Q_L \rightarrow Q$ is defined as: $\beta([u_i]_{\equiv_L}) = q$ with $q_0 \xrightarrow{u_i} q$ in A.

a) Consider the equivalence relation $\equiv_A$. Show that $\text{Index}(\equiv_A) = \text{Index}(\equiv_L)$ holds. With the result $\equiv_A \subseteq \equiv_L$ from the lecture, this implies $\equiv_A = \equiv_L$.

b) Show that β is well-defined.
   Hint: The function β was defined on equivalence classes. You have to show, that β is independent of the choice of the representant $u_1, \dots, u_k$. Let us assume $\hat{u}_i \equiv_L u_i$ and show that $\beta([\hat{u}_i]_{\equiv_L}) = \beta([u_i]_{\equiv_L})$ holds.

c) Show that β is a bijection between Q_L and Q.

d) Show that β is isomorphic. It remains to show, that $\beta(q_{0L}) = q_0$, $\beta(Q_{FL}) = Q_F$ and for all $p, p' \in Q_L$ and $a \in \Sigma$ the property $p \xrightarrow{a}_L p'$ iff $\beta(p) \xrightarrow{a} \beta(p')$ holds.

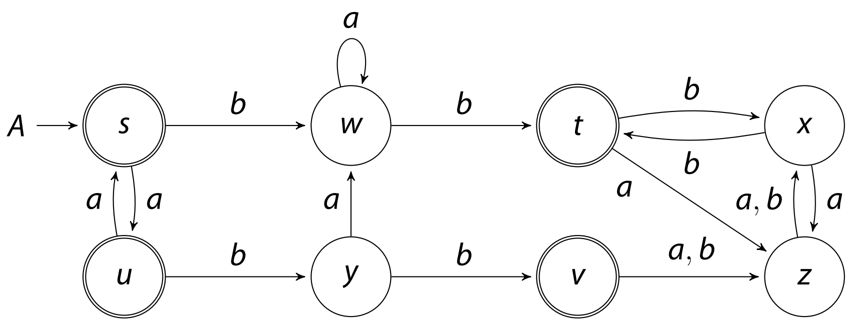
Homework Exercise 7.4: Nerode's right-congruence with non-regular languages [3 points]
Let $\Sigma = \{a, b\}$ be an alphabet. Consider $L = \{ a^n ba^m \mid n, m \in \mathbb{N}, n \geq m \}$. Prove that

$$[a^n]_{\equiv_L} = \{a^n\} \text{ for all } n \in \mathbb{N}$$
$$[a^n.b]_{\equiv_L} = \left\{ a^{n+\ell}.b.a^{\ell} \;\middle|\; \ell \in \mathbb{N} \right\} \text{ for all } n \in \mathbb{N}$$

holds. With infinite congruence classes, L is not regular by the Theorem of Myhill & Nerode.

Find all remaining equivalence classes with respect to $\equiv_L$.

Homework Exercise 7.5: Table-filling algorithm [3 points]
Consider the following DFA A.



Show that A is minimal, by using the table-filling algorithm. Fill cells with 0, if the respective state pair is initially separated, and with the number of the iteration, where that pair is separated for the first time.

Hint: While filling your table, note down in which order you separated a state class, e.g. initially, we separate accepting states from the rest: $\{s, t, u, v\} \not\equiv_A \{w, x, y, z\}$, which allows us to separate $\{s, u\} \not\equiv_A \{t, v\}$ in iteration 1, etc.

Homework Exercise 7.6: Pumping lemma for regular languages [3 points]
Consider $\Sigma = \{a, b\}$. For any word w let $|w|_a$ be the number of occurrences of symbol a in w. $|w|_b$ is defined analogously.
By using the Pumping Lemma, prove that the following languages are not regular.

a) $L_1 = \{ w \in \{a, b\}^* \mid |w|_b + 7 > |w|_a \}$

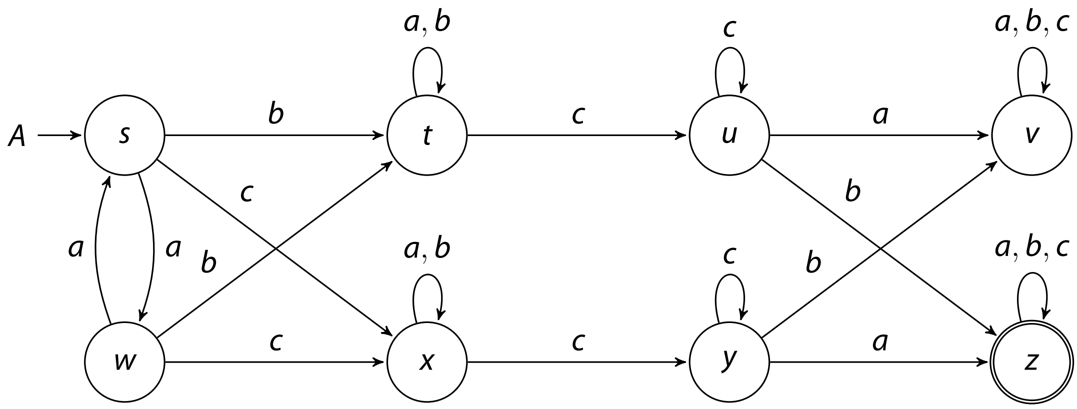
b) $L_2 = \{ a^n b^m \mid n < 42 \text{ or } m < n \}$

c) $L_3 = \{ w \in \{a, b\}^* \mid |w|_a \neq |w|_b \}$

Hint for c): Consider the following: For any given number $n \in \mathbb{N}$, which number is divisible by all numbers $\leq n$?

Exercise 7.7:

Consider the following DFA A. Find its Equivalence-Class-Automaton $A_{\mathcal{L}(A)}$ by Myhill & Nerode by using the Table-Filling algorithm and state all equivalence classes of Nerode's right-congruence.



Exercise 7.8:

Show, that the following languages are not regular, by using the Pumping Lemma.

a) $L_0 := \{ w \in \{a, b\}^* \mid |w|_a \geq |w|_b \}$

b) $L_1 := \{ w \in \{a, b\}^* \mid |w|_a \leq |w|_b$ oder $2|w|_b \leq |w|_a \}$

c) $L_2 := \{ a^n b^m \mid n, m \in \mathbb{N}$ und $(n \neq 1$ oder $\exists \ell \in \mathbb{N}\colon m = \ell^2) \}$.