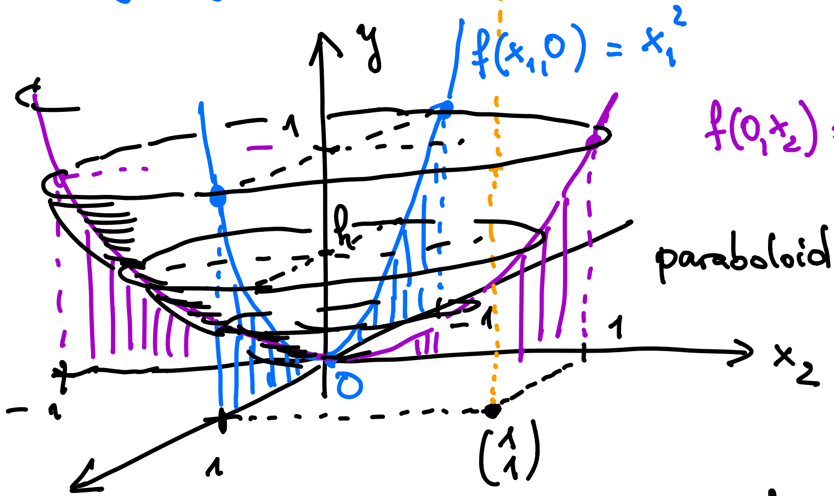
CSE maths refresher course 10th Oct 2024

Multivariate calculus : functions of more than one variable

$f: \mathbb{R}^2 \to \mathbb{R}$

$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto y = f(x_1, x_2)$

e.g. $y = f(x_1, x_2) = x_1^2 + x_2^2$



level lines $h = f(x_1, x_2) = x_1^2 + x_2^2$

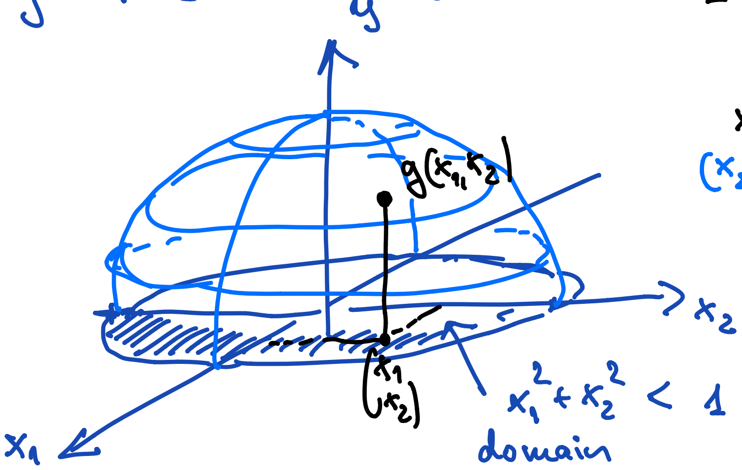
i.e. $\sqrt{h} = \sqrt{x_1^2 + x_2^2}$

circle with radius $\sqrt{h}$

Exp. $y = g(x_1, x_2) = \sqrt{\underbrace{1 - x_1^2 - x_2^2}_{\text{needs to be } \geq 0}} \geq 0$

$y^2 = 1 - x_1^2 - x_2^2$

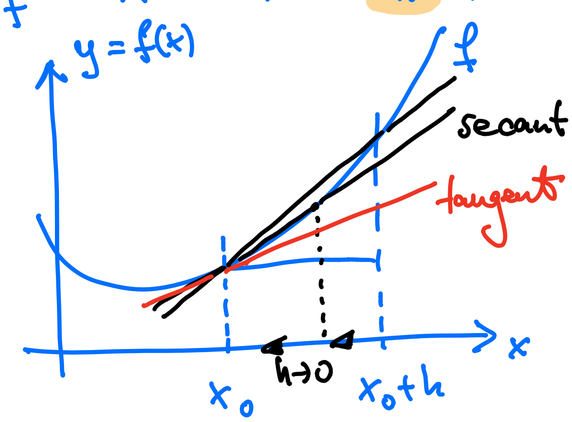
$y^2 + x_1^2 + x_2^2 = 1$ with $y \geq 0$, sphere



Differentiation

partial derivative $\frac{\partial}{\partial x_k} f$: derivative of $f: \mathbb{R}^n \to \mathbb{R}$ w.r.t. the component x_k

rep. $f: \mathbb{R} \to \mathbb{R}$ univariate



$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h} = \frac{d}{dx} f(x_0)$

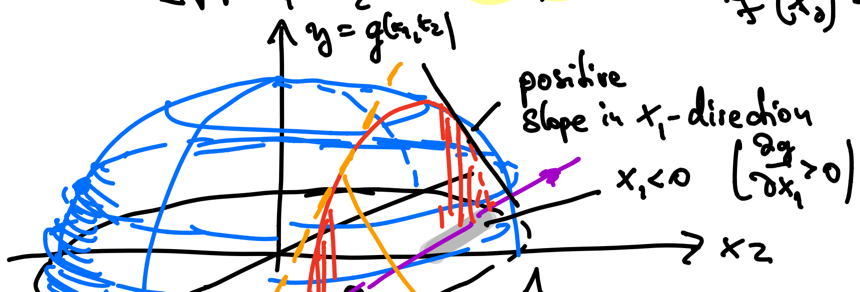
Exp : $f(x_1, x_2) = x_1^2 + x_2^2$

$\frac{\partial}{\partial x_1} f(x_1, x_2) = 2x_1 + 0 = 2x_1$

$\frac{\partial}{\partial x_2} f(x_1, x_2) = 2x_2$

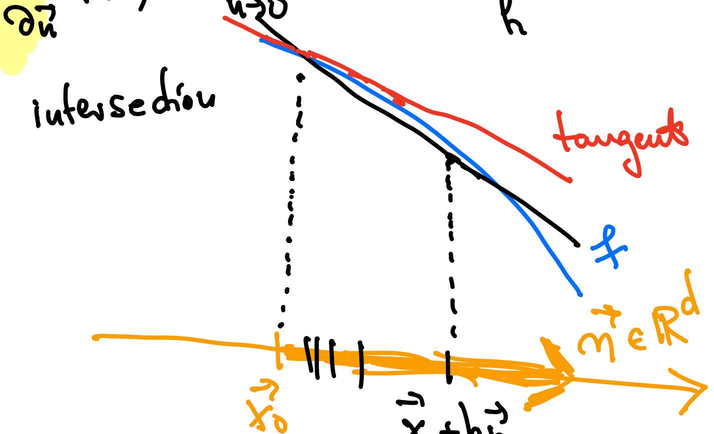
Exp : $g(x_1, x_2) = \sqrt{1 - x_1^2 - x_2^2}$

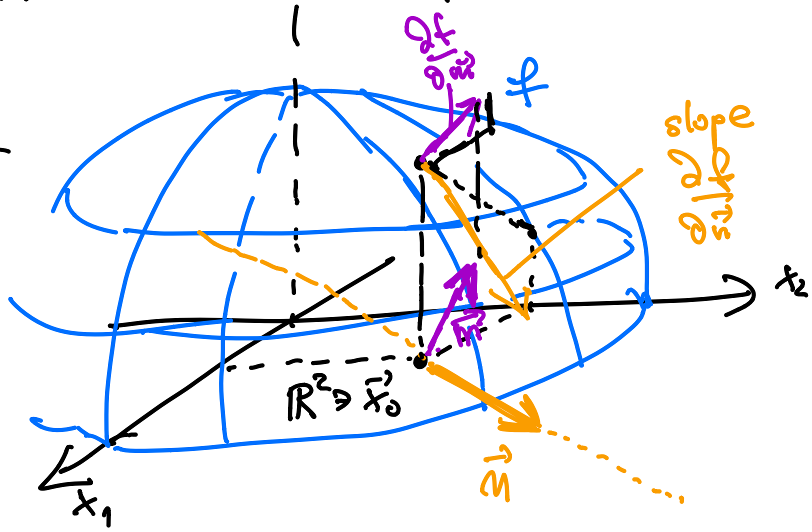
$\frac{\partial}{\partial x_1} g(x_1, x_2) = \frac{-2x_1}{2\sqrt{1 - x_1^2 - x_2^2}} = \frac{-x_1}{\sqrt{1 - x_1^2 - x_2^2}}$



directional derivative $\frac{\partial}{\partial \vec{n}} f(\vec{x}_0)$ with direction $\vec{n} \in \mathbb{R}^d$, $\|\vec{n}\|_2 = 1$

$$\frac{\partial}{\partial \vec{n}} f(\vec{x}_0) = \lim_{h \to 0} \frac{f(\vec{x}_0 + \vec{n}h) - f(\vec{x}_0)}{h}$$





Exp. $f(x_1, x_2) = x_1^2 + x_2^2$, $\vec{x}_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\vec{n} = \frac{1}{5}\begin{pmatrix} 3 \\ 4 \end{pmatrix}$

with $\|\vec{n}\|_2 = \sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2} = 1$

$$\frac{\partial}{\partial \vec{n}} f(\vec{x}_0) = \lim_{h \to 0} \frac{f\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{h}{5}\begin{pmatrix} 3 \\ 4 \end{pmatrix}\right) - f\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)}{h}$$

$$= \lim_{h \to 0} \frac{\left(1 + h\frac{3}{5}\right)^2 + \left(1 + h\frac{4}{5}\right)^2 - 1^2 - 1^2}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{1^2} + 2h\frac{3}{5} + h^2\frac{9}{25} + \cancel{1^2} + 2h\frac{4}{5} + h^2\frac{16}{25} - \cancel{1^2} - \cancel{1^2}}{h}$$

$$= \lim_{h \to 0} \frac{6}{5} + \frac{8}{5} + h\frac{9}{25} + h\frac{16}{25} = \frac{14}{5}$$

Gradient of a function $f : \mathbb{R}^d \to \mathbb{R}$ with $y = f(x_1, x_2, \dots, x_d)$ is in case the function is smooth enough is

$$\text{grad } f = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_d} \end{pmatrix} \in \mathbb{R}^d$$

notation : del - operator $\quad \nabla f = \text{grad } f$

nabla - operator

Theorem: $\frac{\partial}{\partial \vec{n}} f(\vec{x}_0) = \nabla f(\vec{x}_0) \cdot \vec{n}$ if f is smooth enough.

Proof for $d=2$: $\vec{x}_0 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$, direction $\vec{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$ with $n_1^2 + n_2^2 = 1$.

$$\frac{\partial}{\partial \vec{n}} f(\vec{x}_0) = \lim_{h\to 0} \frac{f(\vec{x}_0 + h\vec{n}) - f(\vec{x}_0)}{h}$$

$$= \lim_{h\to 0} \frac{f(x_1 + hn_1, x_2 + hn_2) - f(x_1, x_2)}{h}$$

$$= \lim_{h\to 0} \frac{f(x_1+hn_1, x_2+hn_2) - f(x_1, x_2+hn_2) + f(x_1, x_2+hn_2) - f(x_1,x_2)}{h}$$

$$= n_1 \underbrace{\lim_{h\to 0} \frac{f(x_1+hn_1, x_2+hn_2) - f(x_1, x_2+hn_2)}{hn_1}}_{\frac{\partial f}{\partial x_1}(\vec{x}_0)} + n_2 \underbrace{\lim_{h\to 0} \frac{f(x_1, x_2+hn_2) - f(x_1,x_2)}{hn_2}}_{\frac{\partial f}{\partial x_2}(\vec{x}_0)}$$

$$\frac{\partial}{\partial \vec{n}} f(\vec{x}_0) = n_1 \frac{\partial f}{\partial x_1}(\vec{x}_0) + n_2 \frac{\partial f}{\partial x_2}(\vec{x}_0)$$

$$= \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial f}{\partial x_1}(\vec{x}_0) \\ \frac{\partial f}{\partial x}(\vec{x}_0) \end{pmatrix} = \vec{n} \cdot \nabla f(\vec{x}_0) \qquad \square$$

Exp. $f(x_1, x_2) = x_1^2 + x_2^2$, $\vec{x}_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\vec{n} = \frac{1}{5}\begin{pmatrix} 3 \\ 4 \end{pmatrix}$

$\nabla f = \begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix}$, $\nabla f(\vec{x}_0) = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$, $\nabla f(\vec{x}_0) \cdot \vec{n} = \frac{1}{5}\begin{pmatrix} 2 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \frac{14}{5}$

$\frac{\partial f}{\partial \vec{n}}(\vec{x}_0) = \frac{14}{5} \in \mathbb{R}$

Theorem: The gradient $\nabla f(\vec{x}_0)$ directs into the direction of the steepest ascend / slope of f, i.e. into the direction where $\frac{\partial f}{\partial \vec{n}}(\vec{x}_0)$ is maximal.

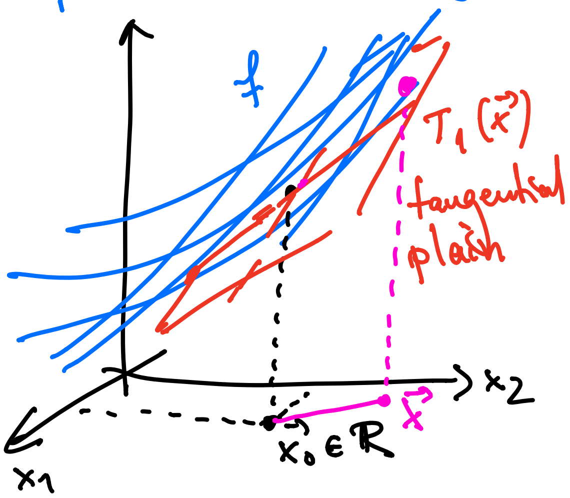
Proof: We ask for which $\vec{n}$ the directional derivative $\frac{\partial f}{\partial \vec{n}}$ is maximal.

$$\frac{\partial}{\partial \vec{n}} f(\vec{x}_0) = \underbrace{\nabla f(\vec{x}_0)}_{\text{independent}} \cdot \vec{n} = \underbrace{\|\nabla f(\vec{x}_0)\|_2}_{\text{fixed}} \cdot \underbrace{\|\vec{n}\|_2}_{=1} \cdot \underbrace{\cos \measuredangle(\nabla f, \vec{n})}_{\text{maximal } = 1}$$

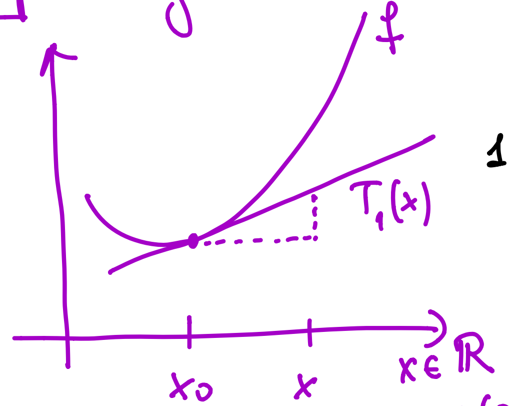
of $\vec{n}$ ... if $\nabla f \parallel \vec{n}$.

i.e. $\frac{\partial}{\partial \vec{n}} f(\vec{x}_0)$ is maximal if ∇f is parallel to $\vec{n}$. $\square$

Equation of the tangential plain of the function $f:\mathbb{R}^2 \to \mathbb{R}$ at $\vec{x}_0 \in \mathbb{R}^2$.



rep. tangent of a univariate function



1. Taylor's polynomial

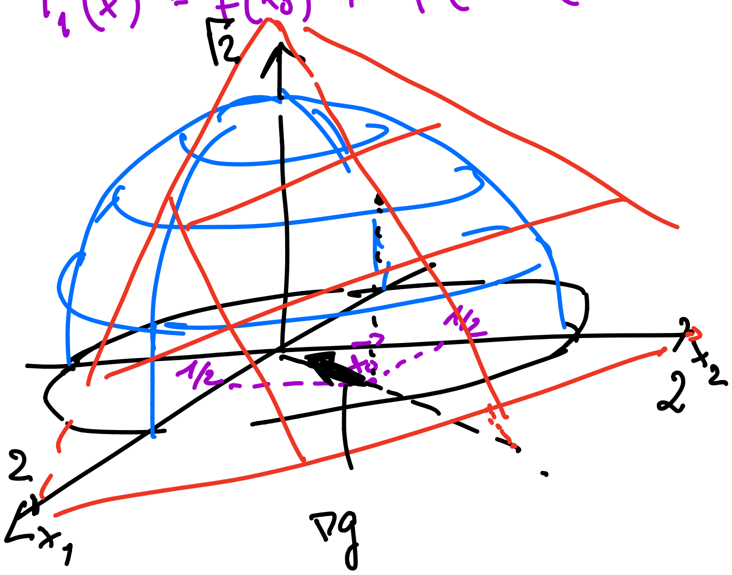
$$T_1(x) = f(x_0) + f'(x_0)\cdot(x - x_0)$$

$$T_1(\vec{x}) = f(\vec{x}_0) + \nabla f(\vec{x}_0)\cdot(\vec{x} - \vec{x}_0)$$

Exp. $y = g(x_1, x_2) = \sqrt{1 - x_1^2 - x_2^2}$

$\vec{x}_0 = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$, $\nabla g = \begin{pmatrix} \frac{-x_1}{\sqrt{\cdots}} \\ \frac{-x_2}{\sqrt{\quad}} \end{pmatrix}$

$$\nabla g(\vec{x}_0) = \begin{pmatrix} \frac{-1/2}{\sqrt{1 - 1/4 - 1/4}} \\ \frac{-1/2}{\sqrt{1 - 1/4 - 1/4}} \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{pmatrix}$$



$$T_1(\vec{x}) = T_1(x_1, x_2) = g\Big(\underbrace{\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}}_{\vec{x}_0}\Big) + \nabla g\Big(\underbrace{\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}}_{\vec{x}_0}\Big)\cdot\Big(\vec{x} - \underbrace{\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}}_{\vec{x}_0}\Big)$$

$$= \frac{1}{\sqrt{2}} + \begin{pmatrix} -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{pmatrix}\cdot\begin{pmatrix} x_1 - 1/2 \\ x_2 - 1/2 \end{pmatrix}$$

$$= \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}(x_1 - 1/2) - \frac{\sqrt{2}}{2}\cdot(x_2 - 1/2) = \frac{\sqrt{2}}{2}\big(1 - (x_1 - 1/2) - (x_2 - 1/2)\big)$$

$$= \frac{\sqrt{2}}{2}(2 - x_1 - x_2) \quad \checkmark$$

Taylor expansion $f:\mathbb{R}^d \to \mathbb{R}$

rep. $f:\mathbb{R} \to \mathbb{R}$, $y = f(x)$

$$T_0(\vec{x}) = f(\vec{x}_0)$$

$$T_1(\vec{x}) = \underbrace{f(\vec{x}_0)}_{\in\mathbb{R}} + \underbrace{\nabla f(\vec{x}_0)}_{\in\mathbb{R}^d} \cdot \underbrace{(\vec{x}-\vec{x}_0)}_{\in\mathbb{R}^d}$$

$$T_2(\vec{x}) = f(\vec{x}_0) + \nabla f(\vec{x}_0)\cdot(\vec{x}-\vec{x}_0) + \dots$$

$$\dots + \frac{1}{2!}\underbrace{\Big[\underbrace{\underbrace{\nabla\nabla f(\vec{x}_0)}_{\substack{\text{Hessian matrix}\\ \in\mathbb{R}^{d\times d}}}\cdot\underbrace{(\vec{x}-\vec{x}_0)}_{\in\mathbb{R}^d}}_{\in\mathbb{R}^d}\Big]\cdot(\vec{x}-\vec{x}_0)}_{\in\mathbb{R}^1}$$

$$T_0(x) = f(x_0)$$

$$T_1(x) = f(x_0) + f'(x_0)\cdot(x-x_0)$$

$$T_2(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2}f''(x_0)(x-x_0)^2$$

$$T_n(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(x_0)\cdot(x-x_0)^k$$

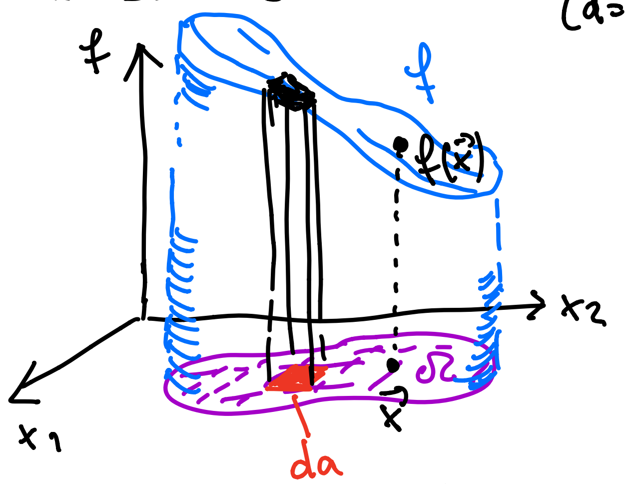
Taylor series

$$T(x) = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(x_0)\cdot(x-x_0)^k$$

Integration of multivariate function

Volume integral

$(d=2)$



function $f : \Omega \subset \mathbb{R}^d \to \mathbb{R}$

$$y = f(x_1, x_2, \dots, x_d)$$

$$f: \vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix} \mapsto y = f(\vec{x})$$

volume under the function f over the domain Ω is

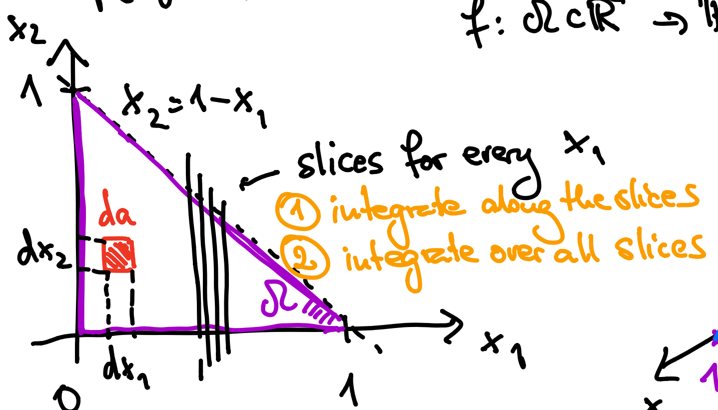
$$\int_\Omega f(\vec{x})\, da$$

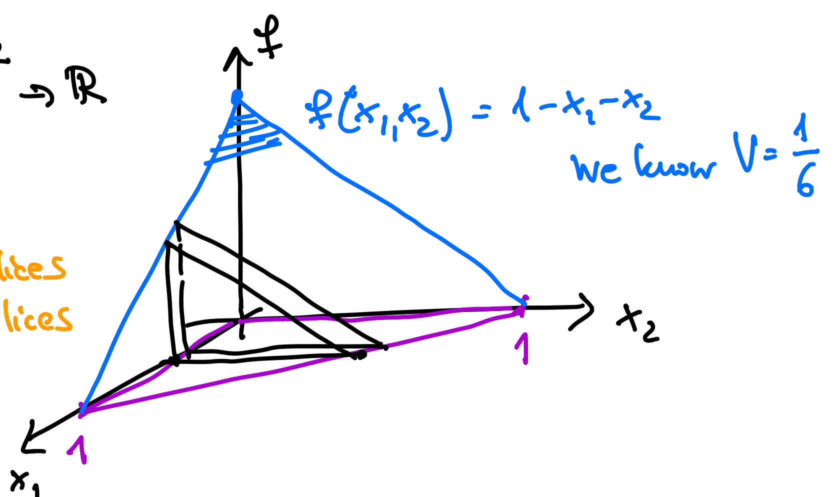
infinitesimal area element

Exp. $\Omega = \{(x_1,x_2)^T \in \mathbb{R}^2 : x_1 \in (0,1),\ x_2 \in (0, 1-x_1)\}$

$$f(\vec{x}) = f(x_1,x_2) = 1 - x_1 - x_2$$

$$f: \Omega \subset \mathbb{R}^2 \to \mathbb{R}$$





"rectangle" $da = dx_1 \cdot dx_2$

constant w.r.t. x_2

$$\int f(\vec{x})\, da = \int_0^1 \left(\int_0^{1-x_1} 1 - x_1 - x_2 \, dx_2 \right) dx_1$$

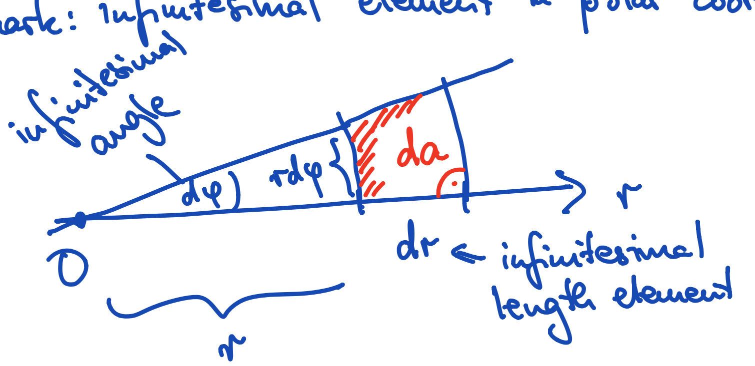
$\Omega \qquad 0 \; \Big(\; 0 \qquad \Big)$

$$= \int_0^1 \Big((1-x_1)x_2 - \frac{x_2^2}{2} \Big|_{x_2=0}^{1-x_1} \Big) \, dx_1 = \int_0^1 \Big(\underbrace{\underbrace{(1-x_1)(1-x_1)}_{①} - \frac{(1-x_1)^2}{2}}_{②} \Big) \, dx_1 =$$

$$= \frac{1}{2} \int_0^1 (1-x_1)^2 \, dx_1 = \frac{1}{2} \cdot (-1) \, \frac{(1-x_1)^3}{3} \Big|_{x_1=0}^{1} = -(-1) \cdot \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$$

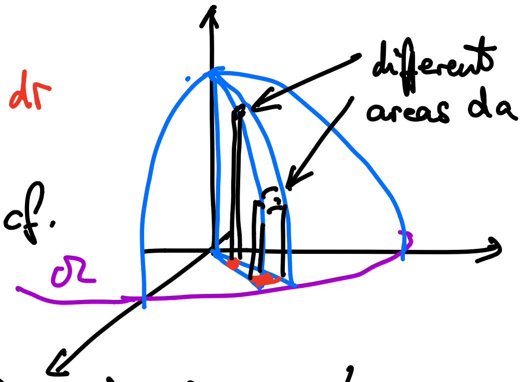
Please try slices into x_1-direction.

Remark: infinitesimal element in polar coordinates



$da = \underline{r} \, d\varphi \, dr$

cf.



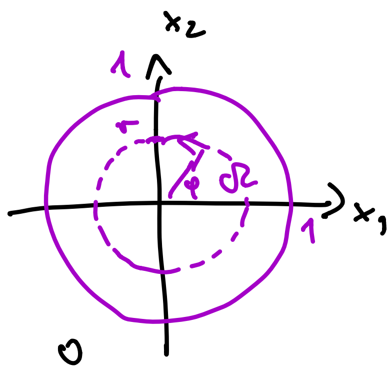
Exp. volume of a half ball, i.e. $\Omega = \left\{ \vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 : \; x_1^2 + x_2^2 < 1 \right\}$ circle

$g(x_1, x_2) = \sqrt{1 - x_1^2 - x_2^2}$

in polar coordinates $\quad \vec{x} = \vec{x}(r, \varphi)$

now $\quad \Omega = \{ \vec{x} = \vec{x}(r,\varphi) \in \mathbb{R}^2 : \; r < 1 \}$ $\qquad r = \sqrt{x_1^2 + x_2^2}$

$g = g(\vec{x}(r,\varphi)) = \sqrt{1 - r^2}$



$$V = \underbrace{\int_\Omega g(\vec{x}) \, da}_{\text{volume integral}} = \underbrace{\int_0^1 \int_0^{2\pi} \underbrace{\sqrt{1 - r^2}}_{g} \cdot \underbrace{\underline{r} \, d\varphi \, dr}_{da}}_{\text{multiple integral}}$$

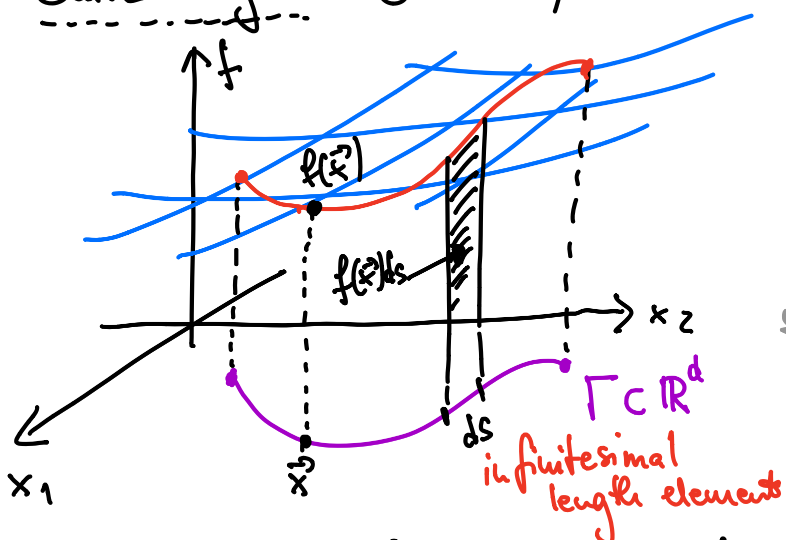
$$= 2\pi \int_{r=0}^{r=1} \underline{r} \sqrt{1 - r^2} \, dr = 2\pi \int_{z=1}^{z=0} \sqrt{z} \, \frac{dz}{-2} = -\pi \int_1^0 \sqrt{z} \, dz = \ldots$$

substitution $\quad z = 1 - r^2, \quad \frac{dz}{dr} = -2r, \quad dz = -2r\,dr$

$$= \pi \int_0^1 z^{1/2} \, dz = \pi \, \frac{2}{3} z^{3/2} \Big|_{z=0}^{1} = \frac{2\pi}{3}$$

Compare the volume of entire ball is $\frac{4\pi}{3}$.

Curve integral (1st kind)



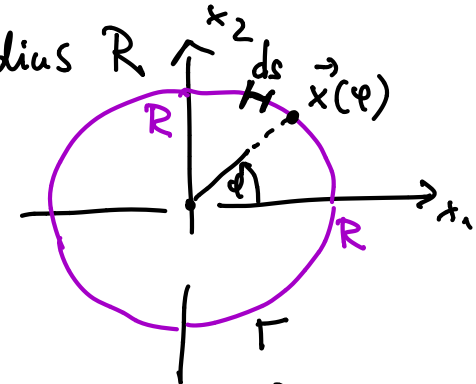
given curve $\Gamma \subset \mathbb{R}^d$

function $f: \mathbb{R}^d \to \mathbb{R}$

area of the flag under f over Γ is

$$\underbrace{\int_\Gamma}_{\text{sum}} \underbrace{f(\vec{x})\, ds}_{\text{area of the stripe}}$$

<u>Exp.</u> $\Gamma = \{\vec{x} \in \mathbb{R}^2 : \|\vec{x}\|_2 = R\}$ circle with radius R



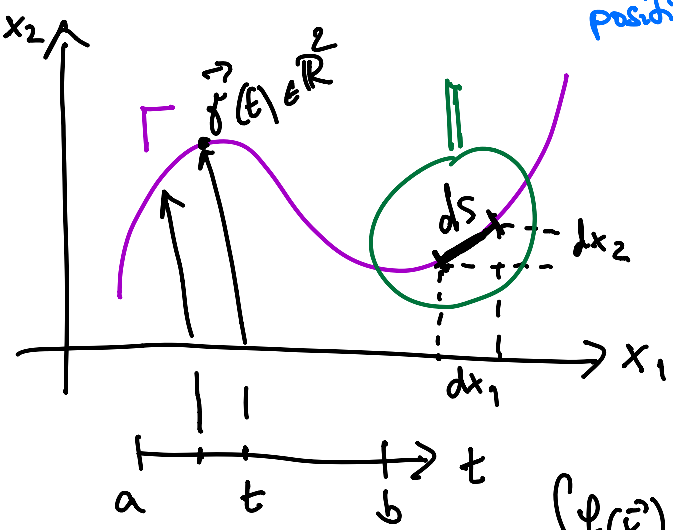
parameterise Γ by $\varphi \in [0, 2\pi) \Rightarrow ds = R\, d\varphi$

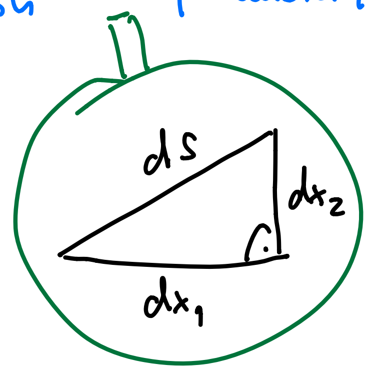
$$\underbrace{\int_\Gamma f(\vec{x})\, ds}_{\text{curve integral}} = \underbrace{\int_0^{2\pi} f(\vec{x}(R,\varphi))\, R\, d\varphi}_{\text{standard 1d integral over } \varphi}$$

e.g. $f(\vec{x}) = 1 \Rightarrow \ldots \int_0^{2\pi} 1\, R\, d\varphi = 2\pi R$, cf. circumference of the circle

$\uparrow$ flag of height 1.

We determine ds for $\Gamma = \{\underbrace{\vec{\gamma}(t)}_{\text{position}} \in \mathbb{R}^2 : \underset{\substack{\uparrow \\ \text{parameter / time}}}{t} \in [a,b]\}$





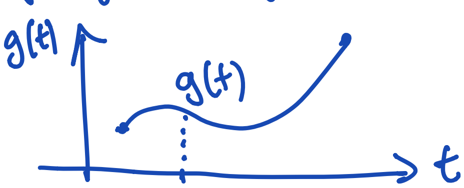
$$ds = \sqrt{(dx_1)^2 + (dx_2)^2} \quad \text{Pythagoras}$$

$$\Rightarrow \frac{ds}{dt} = \sqrt{\Big(\underbrace{\frac{dx_1}{dt}}_{\gamma_1'(t)}\Big)^2 + \Big(\underbrace{\frac{dx_2}{dt}}_{\gamma_2'(t)}\Big)^2}$$

i.e. $ds = \sqrt{\gamma_1'(t)^2 + \gamma_2'(t)^2}\, dt$

$$\int_\Gamma f(\vec{x})\, ds = \int_a^b f(\underbrace{\gamma_1(t), \gamma_2(t)}_{=\vec{x}}) \sqrt{\gamma_1'(t)^2 + \gamma_2'(t)^2}\, dt$$

if $\vec{\gamma}(t) = \begin{pmatrix} t \\ g(t) \end{pmatrix}$ i.e. $x_1 = t$, $x_2 = g(x_1) = g(t)$



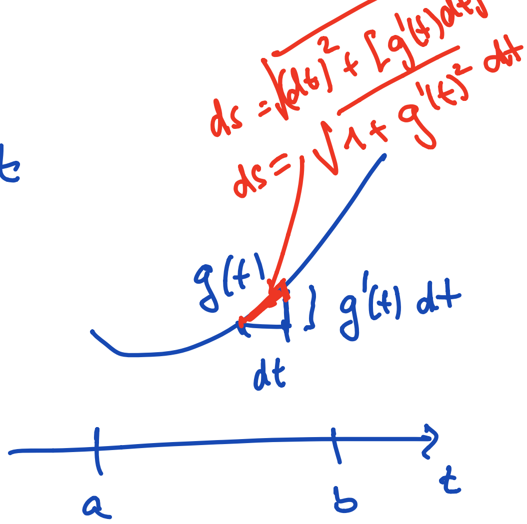
$$\int_\Gamma f(\vec{x})\, ds = \int_a^b f(t, g(t)) \sqrt{1 + g'(t)^2}\, dt$$

in particular for the length of a curve

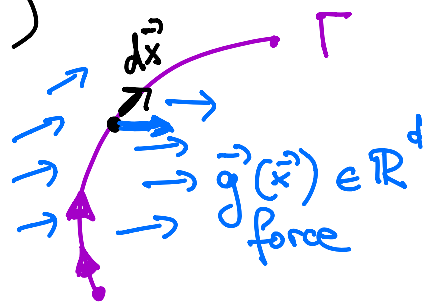
$$|\Gamma| = \int_\Gamma 1\, ds = \int_a^b \underbrace{\sqrt{\gamma_1'(t)^2 + \gamma_2'(t)^2}}_{\text{speed alongst } \Gamma}\, dt$$

in the case $\vec{\gamma}(t) = \begin{pmatrix} t \\ g(t) \end{pmatrix}$, $t \in [a,b]$

$$|\Gamma| = \int_a^b \sqrt{1 + g'(t)^2}\, dt$$



<u>Work integral</u> (curve integral of 2nd kind)



given curve Γ

force field $\vec{g}: \mathbb{R}^d \to \mathbb{R}^d$

ask for the work of $\vec{g}$ alongst Γ

work integral $\quad W = \int_\Gamma \underbrace{\vec{g} \cdot d\vec{x}}_{dW}$ with directional element $d\vec{x} = \begin{pmatrix} dx_1 \\ dx_2 \\ \vdots \\ dx_d \end{pmatrix}$

comp. $\dfrac{dW}{dt} = \vec{g} \cdot \dfrac{d\vec{x}}{dt}$

power = force · velocity

$$W = \int_\Gamma \begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_d \end{pmatrix} \cdot \begin{pmatrix} dx_1 \\ dx_2 \\ \vdots \\ dx_d \end{pmatrix} = \int_\Gamma g_1(\vec{x})\, dx_1 + \dots + g_d(\vec{x})\, dx_d \,.$$

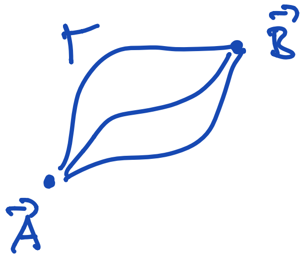
parameterization $\vec{x} = \vec{\gamma}(t)$, $t \in [a,b]$, $x_1 = \gamma_1(t)$, $\dfrac{dx_1}{dt} = \gamma_1'(t)$

$$W = \int_\Gamma \vec{g} \cdot d\vec{x} = \int_a^b \left(g_1(\vec{x}) \frac{dx_1}{dt} + \dots + g_d(\vec{x}) \frac{dx_d}{dt} \right) dt$$

$$= \int_a^b \underbrace{\underbrace{\vec{g}(\vec{x}(t))}_{\text{force}} \cdot \underbrace{\vec{\gamma}'(t)}_{\text{velocity}}}_{\text{power}}\, dt \,.$$

Last: conversative force $\vec{g}(\vec{x}) = \nabla \phi(\vec{x})$ with $\underbrace{\text{potential } \phi(\vec{x})}_{\text{potential energy}}$

convention $-\nabla\phi$



then the work on a path Γ connecting $\vec{A}$ and $\vec{B}$ depends only on $\vec{A}$ and $\vec{B}$ and not on the particular curve Γ.

potential ...

$$\int_\Gamma \vec{g} \cdot d\vec{x} = \phi(\vec{B}) - \phi(\vec{A})$$

Proof: $\Gamma = \{ \vec{\gamma}(t) \in \mathbb{R}^d, t \in [a,b] \}$

with $\vec{\gamma}(a) = \vec{A}$ and $\vec{\gamma}(b) = \vec{B}$.

$$W = \int_\Gamma \vec{g}(\vec{x}) \cdot d\vec{x} = \int_a^b \vec{g}(\vec{\gamma}(t)) \cdot \vec{\gamma}\,'(t)\, dt$$

$$= \underbrace{\int_a^b \overset{\text{outer derivative}}{\nabla \phi(\vec{\gamma}(t))} \cdot \overset{\text{inner derivative}}{\vec{\gamma}\,'(t)}\, dt}_{\text{chain rule}} = \overset{\text{integral over the derivative}}{\int_a^b \frac{d}{dt} \phi(\vec{\gamma}(t))\, dt}$$

$$= \phi(\vec{\gamma}(b)) - \phi(\vec{\gamma}(a)) = \phi(\vec{B}) - \phi(\vec{A}).$$

□

Corollary: The work of $\vec{g} = \nabla \phi$ over a closed loop is zero.

By the way $\nabla \times \vec{g} = \vec{0}$, no curls.