CSE : maths refresher course 7th Oct 24

Univariate functions : sketches, plots, properties etc.

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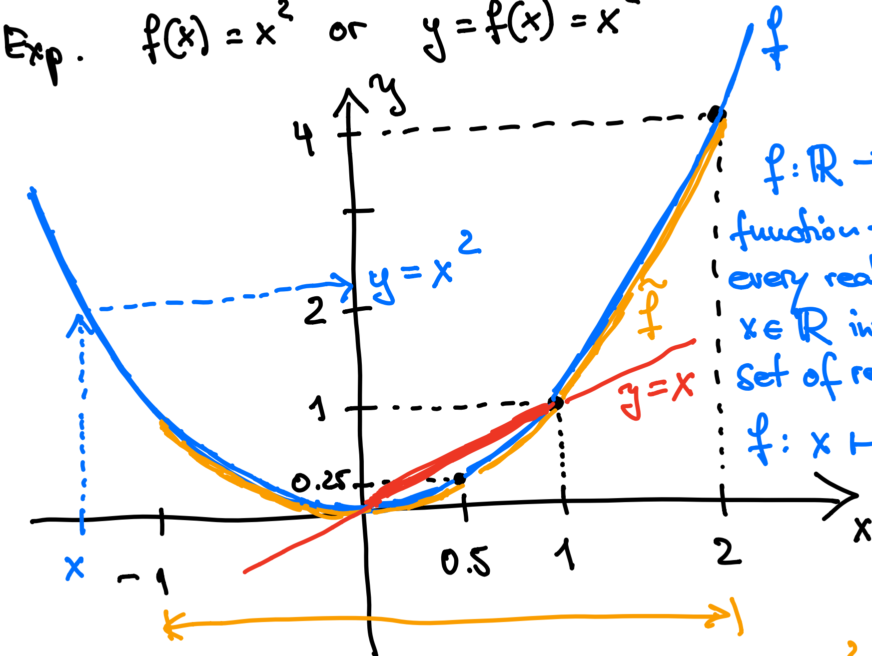
Exp. $f(x) = x^2$ or $y = f(x) = x^2$



$f: \mathbb{R} \to \mathbb{R}$
function f maps every real number $x \in \mathbb{R}$ into the set of real numbers $\mathbb{R}$
$f: x \mapsto x^2 = y$

Exp. $\tilde{f}: \underbrace{[-1,2]}_{\text{interval domain}} \to \underbrace{\mathbb{R}}_{\text{codomain}}$ with $f: x \mapsto x^2$

$[-1,2] = \{x \in \mathbb{R}: -1 \leq x \leq 2\}$
closed interval

Exp. equation $x^2 = x$, solutions?
$x_1 = 0$, $x_2 = 1$

compare $x^2 = x \quad | : x \quad ! \, x \neq 0$
$x_2 = 1$ check $x_1 = 0$

Exp. inequality $x^2 < x$
set $\mathcal{L} = \{x \in \mathbb{R} : 0 < x < 1\}$
$= (0,1)$ open interval

compare $x^2 < x \quad | : x$
case distinction

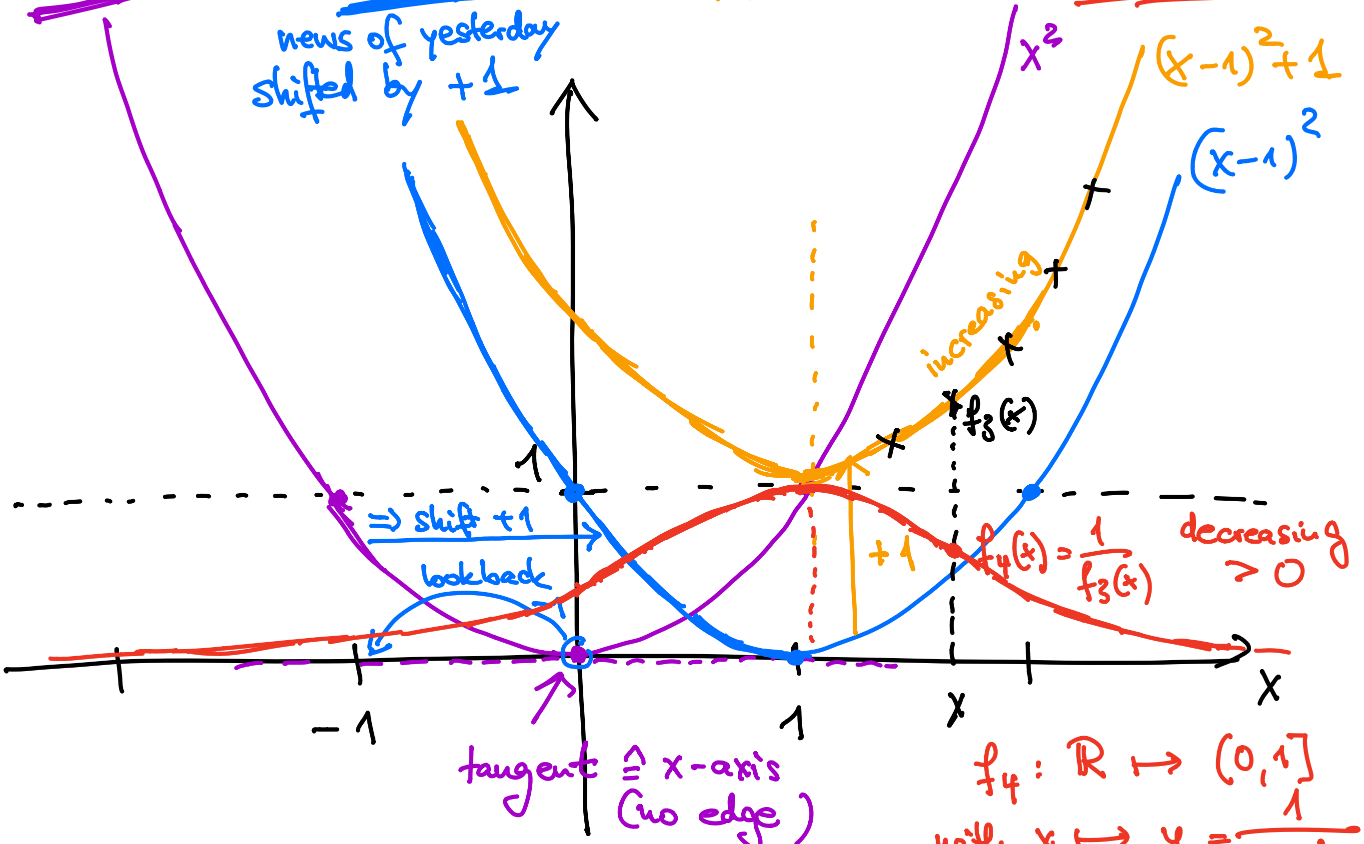
$x < 0$	$x > 0$	extra $x = 0$
$x > 1$ contradiction no x	$x < 1$ $\Rightarrow x \in (0,1)$	$0^2 \not< 0$

Exp. $f(x) = x^3 - 2x^2 + \frac{1}{2}x + 1$

← x^3 grows faster than x^2 for $x \to \infty$

sketches !

Exp. $f_1(x) = x^2$, $f_2(x) = (x-1)^2$, $f_3(x) = (x-1)^2 + 1$, $f_4(x) = \dfrac{1}{(x-1)^2+1}$



$f_4 : \mathbb{R} \mapsto (0,1]$

with $x \mapsto y = \dfrac{1}{(x-1)^2+1}$

here $f(\mathbb{R}) = (0,1]$

Exp. $f(x) = 2^x$, $f: \mathbb{R} \to \mathbb{R}$, $x \mapsto y = 2^x$

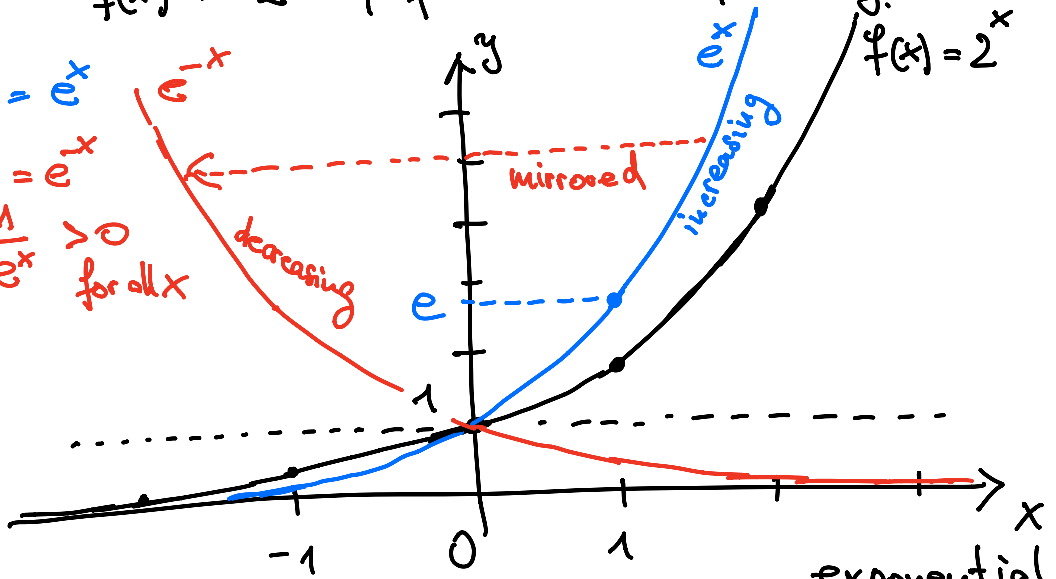
cf. $2^{-1} = \frac{1}{2}$

$2^{-2} = \frac{1}{4}$

$g(x) = e^x$

$h(x) = e^{-x}$

$= \frac{1}{e^x} > 0$ for all x



Euler's number

$e = \lim\limits_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$

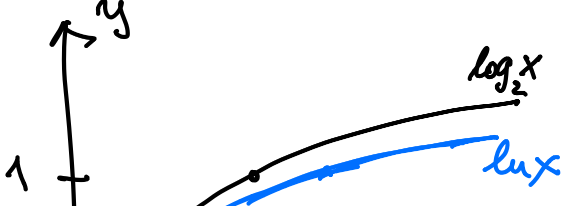
$e \approx 2.718 \ldots$

$\frac{d}{dx} e^x = e^x$

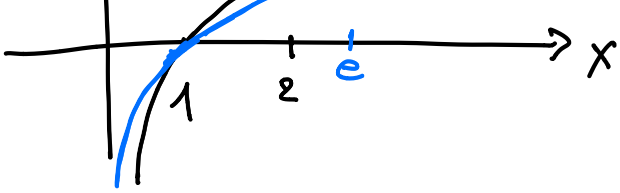
exponential functions

inverse functions of exponential function : logarithmic function

$2^3 = 8$, $\sqrt[3]{8} = 2$, $\log_2 8 = 3$



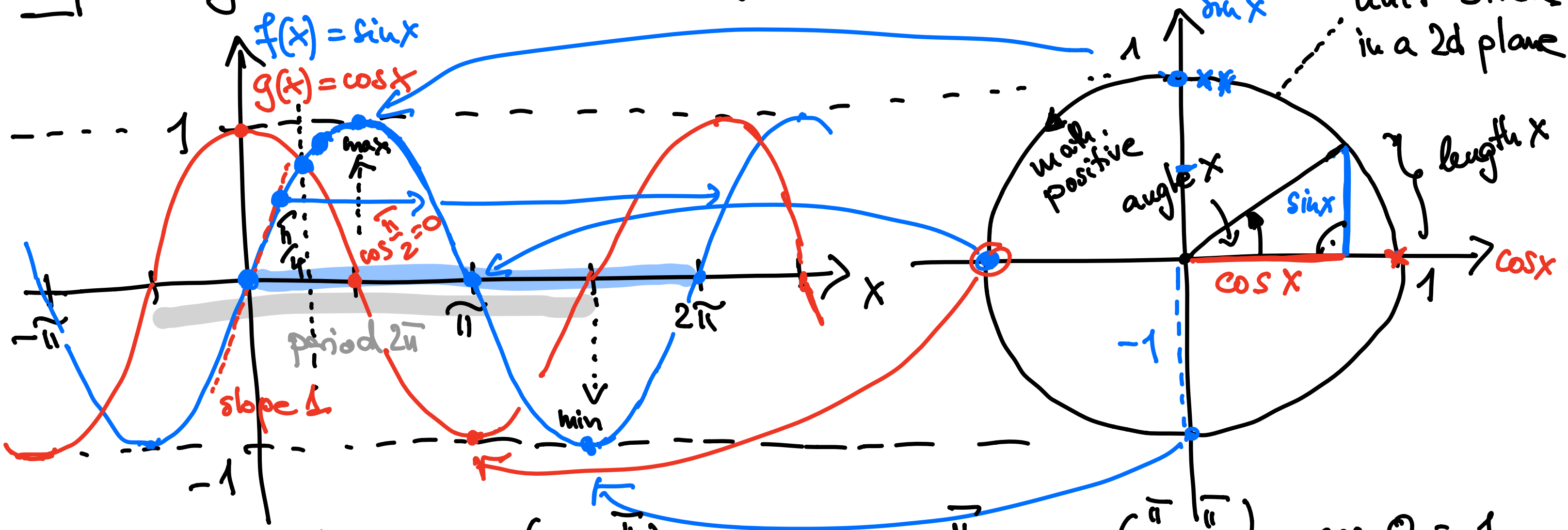
$y = \log_2 x \iff 2^y = x$



$y = \log_e x = \ln x \iff e^y = x$

logarithmus naturalis

Exp. trigonometric functions, e.g. $f(x) = \sin x$, $g(x) = \cos x$



$\sin x = \cos\left(x - \frac{\pi}{2}\right)$ e.g. $\sin\frac{\pi}{2} = \cos\left(\frac{\pi}{2} - \frac{\pi}{2}\right) = \cos 0 = 1$

symmetry $\sin x = \cos\left(\frac{\pi}{2} - x\right)$ cf. $\sin(-x) = -\sin x$ "odd"

$\cos(-x) = \cos x$ "even"

comp. $\frac{d}{dx}\sin x = \cos x$

rem. $\sin x = 0$ zeros $x_k = k\pi$, $k \in \mathbb{Z}$, integers

$\cos x = 0$ zeros $x_k = \frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$

Some values

x	$\sin x$	$\cos x$
0	$0 = \frac{\sqrt{0}}{2}$	1
$\pi/6$	$\frac{1}{2} = \frac{\sqrt{1}}{2}$	$\frac{\sqrt{3}}{2}$
$\pi/4$	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$
$\pi/3$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$\pi/2$	$1 = \frac{\sqrt{4}}{2}$	0

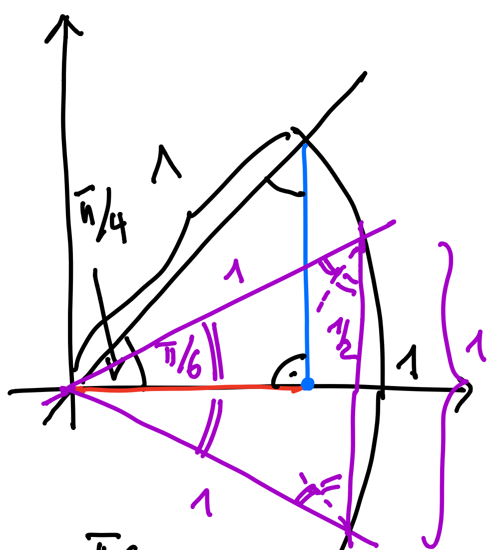
$x = \pi/4$

$\sin x = \cos x$

Pythagoras

$\sin^2 x + \cos^2 x = 1$

$\sin^2 x = \cos^2 x = \frac{1}{2}$



$x = \pi/6$

$\cos^2\frac{\pi}{6} = 1 - \sin^2\frac{\pi}{6} = \frac{3}{4}$

Complex numbers

$x^2+1=0$ has no solution in $\mathbb{R}$

imaginary unit i fulfills $\boxed{i^2=-1}$

immediately $(-i)^2=-1$

complex number $z = x + iy \quad , x,y \in \mathbb{R}$

cf. pairs (x,y) of real numbers

operations $(x+iy) \underset{\text{in } \mathbb{C}}{\pm} (u+iv) = (x \underset{\text{in } \mathbb{R}}{\pm} u) + i(y \underset{\text{in } \mathbb{R}}{\pm} v)$

$(x+iy) \underset{\text{in } \mathbb{C}}{\cdot} (u+iv) = x \underset{\text{in } \mathbb{R}}{\cdot} u + i x\cdot v + i y \cdot u + \underbrace{i^2}_{-1} y \cdot v$

$= xu - yv + i(xv+yu) \in \mathbb{C}$ ← set / field of complex numbers

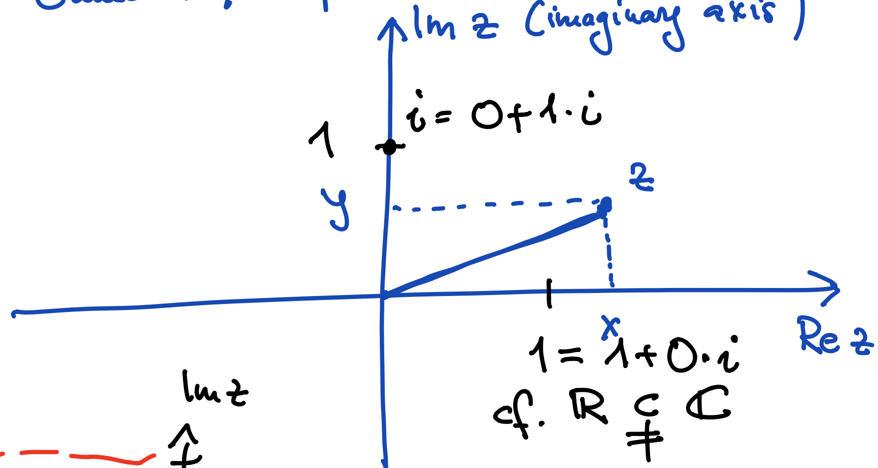
pair $(xu-yv \, , \, xv+yu)$

$z = x+iy$ complex number

$x = \mathrm{Re}\, z$ real part

$y = \mathrm{Im}\, z$ imanigary part

Gaussian / complex plane



cf. $\mathbb{R} \subsetneq \mathbb{C}$

Exp. $z_1 = 2+i \quad , z_2 = -1+3i$

$z_1+z_2 = (2+i)+(-1+3i) = 1+4i$

$2z_1 = 2(2+i) = 4+2i$

$z_1 \cdot z_2 = (2+i)(-1+3i)$

$= -2 - i + 6i + 3i^2$

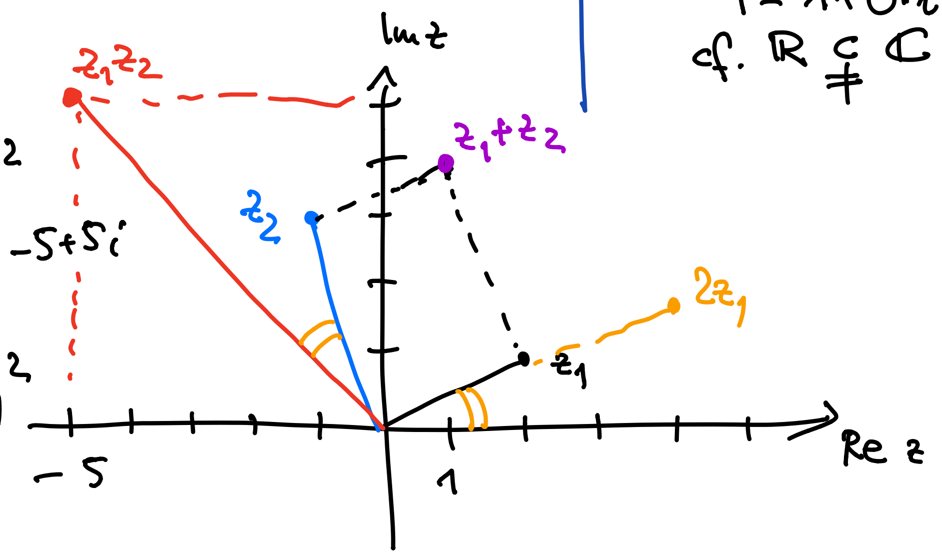
$= -2+5i-3 = -5+5i$

$z_2^2 = (-1+3i)^2$

$= (-1)^2 + 2\cdot(-1)3i + (3i)^2$

$= 1 - 6i + 9\underbrace{i^2}_{-1}$

$= -8-6i$



remember: make the denominator rational

$\frac{1}{2+\sqrt{2}} = \frac{1}{2+\sqrt{2}} \cdot \frac{2-\sqrt{2}}{2-\sqrt{2}} = \frac{2-\sqrt{2}}{2^2-\sqrt{2}^2}$

$= \frac{2-\sqrt{2}}{2}$ ← rational

division in $\mathbb{C}$

$\frac{x+iy}{u+iv} = \frac{x+iy}{u+iv} \cdot \frac{u-iv}{u-iv} = \frac{xu+yv+i(yu-xv)}{\underbrace{u^2+v^2}_{\text{real}}} \in \mathbb{C}$

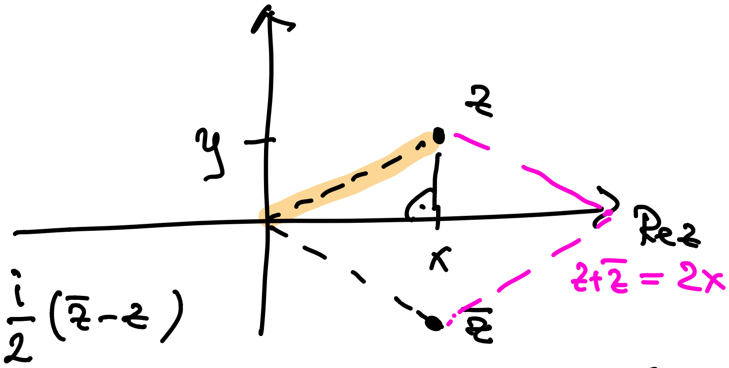
Exp. $\frac{z_1}{z_2} = \frac{2+i}{-1+3i} = \frac{(2+i)(-1-3i)}{(-1+3i)(-1-3i)} = \frac{-2+3+i(-1-6)}{(-1)^2+3^2} = \frac{1}{10} - i\frac{7}{10} \in \mathbb{C}$

Notation $z = x + iy \in \mathbb{C}$

conjugate number $\bar{z} = x - iy$

absolute value $|z| = \sqrt{x^2 + y^2}$

! distance to $0 \in \mathbb{C}$



$$\mathrm{Re}\, z = \frac{1}{2}(z + \bar{z}), \quad \mathrm{Im}\, z = \frac{1}{2i}(z - \bar{z}) = \frac{i}{2}(\bar{z} - z)$$

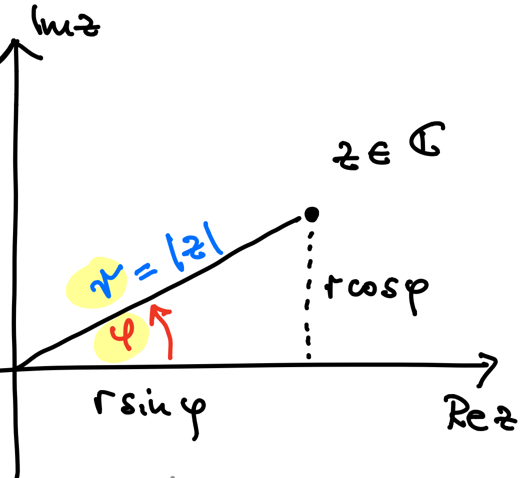
$$|z|^2 = z \cdot \bar{z} = (x+iy)(x-iy) = x^2 + \underbrace{ixy - ixy}_{=0} - (iy)^2 = x^2 + y^2$$

$$\frac{1}{z} = \frac{\bar{z}}{z \cdot \bar{z}} = \frac{\bar{z}}{|z|^2}$$

Complex numbers in polar coordinates

$z = x + iy$ is described by the radius $r = |z| = \sqrt{x^2+y^2}$ and the angle φ

$$z = r(\cos\varphi + i \sin\varphi)$$



Euler's identity $\boxed{e^{i\varphi} = \cos\varphi + i \sin\varphi}$

$$z = r e^{i\varphi}$$

$e^{i\varphi}$: exponential with complex argument; $\cos\varphi + i\sin\varphi$: trigonometric

cf. Taylor's series

$$\sin\varphi = \varphi - \frac{\varphi^3}{3!} + \frac{\varphi^5}{5!} - \frac{\varphi^7}{7!} \pm \dots$$

$$\cos\varphi = 1 - \frac{\varphi^2}{2!} + \frac{\varphi^4}{4!} - \frac{\varphi^6}{6!} \pm \dots$$

$$e^{w} = 1 + w + \frac{w^2}{2!} + \frac{w^3}{3!} + \frac{w^4}{4!} + \dots$$

$$e^{i\varphi} = 1 + i\varphi + i^2\frac{\varphi^2}{2!} + i^3\frac{\varphi^3}{3!} + i^4\frac{\varphi^4}{4!} + \dots$$

$$= \underbrace{1 - \frac{\varphi^2}{2!} + \frac{\varphi^4}{4!} \mp \dots}_{\cos\varphi} + i \cdot \underbrace{\left(\varphi - \frac{\varphi^3}{3!} \pm \dots\right)}_{\sin\varphi}$$

- product and quotient get easy in polar coordinates

$$z_1 = r_1 e^{i\varphi_1}, \quad z_2 = r_2 e^{i\varphi_2}$$

$$z_1 \cdot z_2 = r_1 \cdot r_2 \cdot e^{i(\varphi_1 + \varphi_2)}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\varphi_1 - \varphi_2)}$$

rem: $e^{i(\varphi+\psi)} = \cos(\varphi+\psi) + i \sin(\varphi+\psi)$

$$= e^{i\varphi} \cdot e^{i\psi} = (\cos\varphi + i \sin\varphi)(\cos\psi + i\sin\psi)$$

$$= \cos\varphi\cos\psi - \sin\varphi\sin\psi + i(\sin\varphi\cos\psi + \cos\varphi\sin\psi)$$

equating coefficient $\rightsquigarrow$ $\cos(\varphi+\psi) = \cos\varphi\cos\psi - \sin\varphi\sin\psi$

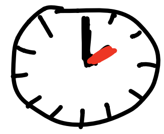
$\sin(\varphi+\psi) = \sin\varphi\cos\psi + \cos\varphi\sin\psi$

Roots of complex numbers

rem $\sqrt{9} = 3$ because $3^2 = 9$ but $(-3)^2 = 9$

every z with $z^n = b$, $n \in \mathbb{N}, n > 0$ is called a n-th complex root of b

-3 and 3 are 2nd roots of $9 \in \mathbb{C}$ because $(-3)^2 = 9$ and $3^2 = 9$



2 o'clock
14 –"–
26 –"–
38 –"–

$b = s e^{i\psi}$, ansatz for root $z = r e^{i\varphi}$

$z^n = b \Rightarrow (r e^{i\varphi})^n = r^n e^{in\varphi} = s e^{i\psi}$

$\Rightarrow r^n = s \in \mathbb{R}$ and $n\varphi$ and ψ describe the same direction

$r = \sqrt[n]{s}$

$n\varphi = \psi + k \cdot 2\pi$ with $k \in \mathbb{Z}$

$\varphi_k = \frac{\psi}{n} + k\frac{2\pi}{n}$, $k = 0, 1, \ldots, n-1$

$z_k = \sqrt[n]{s}\, e^{i\left(\frac{\psi}{n} + k\frac{2\pi}{n}\right)}$, $k = 0, 1, \ldots, n-1$

$\leadsto$ n n-th roots of b

check $e^{2\pi i} = \cos 2\pi + i \sin 2\pi = 1 + 0i = 1$

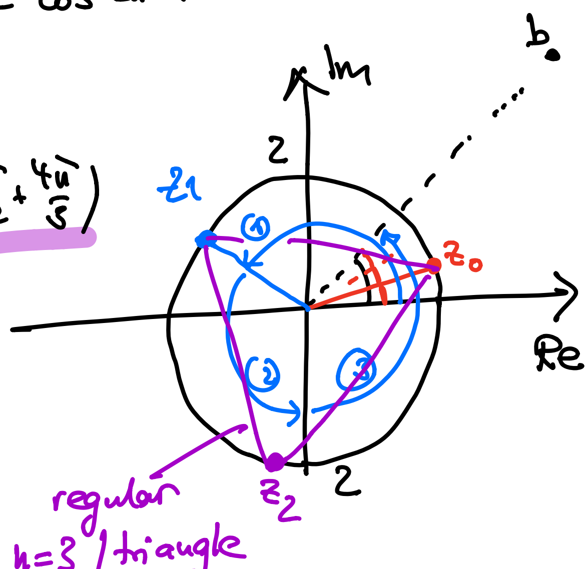
Exp. $b = 8 e^{i\frac{\pi}{4}}$, $n = 3$

i.e. $b = 8$, $\psi = \frac{\pi}{4}$

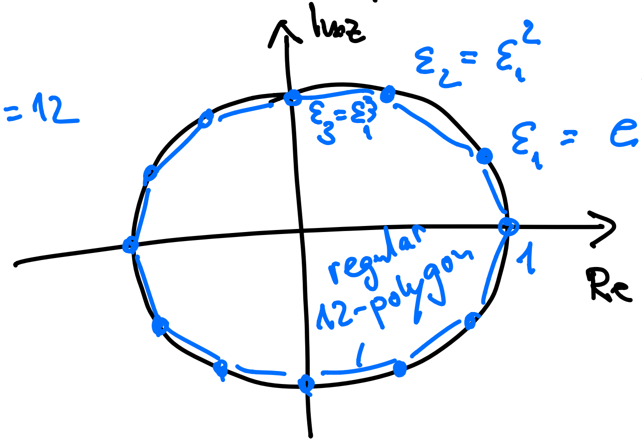
$z_0 = 2 e^{i\frac{\pi}{12}}$, $z_1 = 2 e^{i\left(\frac{\pi}{12} + \frac{2\pi}{3}\right)}$, $z_2 = 2 e^{i\left(\frac{\pi}{12} + \frac{4\pi}{3}\right)}$

$z_1^3 = 2^3 e^{i\left(\frac{\pi}{4} + 2\pi\right)}$

$= 8 e^{i\frac{\pi}{4}} \underbrace{e^{i2\pi}}_{=1}$ ✓



Exp. $b = 1 = 1e^{i0}$, $z_k = 1 e^{i\left(\frac{0}{n} + \frac{2\pi}{n}k\right)}$, $k = 0, 1, \ldots, n-1$



unit roots $\varepsilon_1, \varepsilon_2 = \varepsilon_1^2, \ldots$

$\ldots, \varepsilon_{11} = \varepsilon_1^{11}, \varepsilon_{12} = 1$.